Price of Anarchy in Network Routing with Class based capacity Guarantees

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***Abstract*—In this paper, we consider the inefficiency of distributed routing in a network of parallel links with class-based traffic. Network link behavior is modeled by the M/M/1-GPS queue (i.e. when links use General Processor Sharing(GPS) scheduling scheme to serve packets). Each traffic type is guaranteed a minimum capacity rate on each link using GPS scheduling. We show under specific demand conditions that, among multiple equilibria the worst-case Nash equilibrium occurs when each class dispatcher utilizes all the links to fulfill its demand. Using this fact, we give an upper bound on the Price of Anarchy (PoA). Our results also indicate that, while the price of selfish behavior can be unbounded in a specific demand setting, there exist demand regimes where the bound on PoA is reasonable. These results also apply to the resource allocation or load balancing applications in the processor sharing systems.**

I. INTRODUCTION

The provision of Quality of Service (QoS) is an influential topic and has become an increasingly challenging in the design of network routing protocols. One popular approach to maintain QoS in the network routing is via service differentiation of several traffic classes in the network. Traffic classes often have stringent QoS requirements such as bandwidth, delay, jitter, loss rate and each traffic type should be treated separately according to its application needs. QoS allows network designers to categorize the traffic into classes and serve them according to their inherent QoS requirements. For example, multimedia traffic like video and audio need to meet certain latency bound. There has been extensive research on the methods of providing QoS to different classes of data in the network routing. IntServ[1] and DiffServ [2] are two main approaches emerged to tackle the problem. One of the effective approach to ensure QoS is the scheduling policy that a network link incorporates to serve its arrival packets. This technique is easy to implement in the core of the network and has low cost. Note that, not all scheduling schemes are necessarily QoS inherent. For instance, First Come First Serve (FCFS) scheduling approach does not differentiate between different class of data. Among promising scheduling mechanisms, Generalized Processor Sharing (GPS) which was first studied by Parekh and Gallanger [3], has QoS features and has attracted numerous attention from both the academia and the industry [4]. In this paper, we use GPS scheduling scheme to implement QoS in our network routing model. We are particularly interested in the bandwidth reservation aspect of GPS scheduler, which ensures that all classes have a minimum capacity guaranteed in the network.

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In large-scale communication networks such as the Internet, number of traffic sources (i.e. the ISP), transfer their packets using several routing paths. In such distributed routing approach, each data source acts as an independent rational agent participating in a non-cooperative routing game in the context of game theory[5]. A number of ISPs (players) selfishly attempt to select the best routing paths according to some network performance metrics in respond to decision of other players. This interaction between agents eventually leads to a stable point called the "Nash equilibrium" point. At Nash equilibrium all agents are satisfied with their chosen routing paths and none of them find it preferable to use other subset of routes to send their data [6]. Characteristics of the Nash equilibrium point have been analyzed extensively in different economic and networking models[7]–[9]. In a non-cooperative game, it is already known that, the selfish behavior of the players could cause performance degradation when compared with optimal solution which requires a central decision maker. This inefficiency, known as *Price of Anarchy* (PoA)[10], is due to the negative externality that a player imposes on others when it shares a resource with them. Having a bounded PoA justifies the selfish routing scheme in the scenarios where central approach implementation is costly or impractical.

In most networks, PoA is tightly related to the network link cost function which is affected by QoS mecha-

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nism. Despite the extensive literature on analyzing PoA for different cost functions[10], [11], only a few have assumed the QoS on their model and heterogeneous traffic[11]–[13]. In this paper, we aim to analyze the selfish behavior of ISPs in a routing game when QoS is incorporated by using GPS scheduling on each links. Our goal to study GPS in selfish routing is two folded. First, we want to understand how using GPS scheduling could affect inefficiency of selfish routing. Second, we wonder if GPS characteristic and its related parameters could help the network operators to design an efficient network. There are two main contributions in this paper. First, Specifically, we characterize the existence of mixed equilibrium. Second, we show that the worst case Nash equilibrium always occurs at particular flow assignment assuming specific condition on demands or capacities. Finally, we derive a closed form upper bound on PoA under GPS scheduling discipline in a network of parallel links.

II. RELATED WORK

A. Selfish routing and PoA

The efficiency of selfish routing have been extensively studied through last decade by computer scientist. Initially, Papadimitriou studied the inefficiency of the Nash equilibrium for a network of parallel links [14]. Later, Roughgarden *et al.* in [10] proved that PoA in any network with continuous and monotone cost function is bounded tightly by a value called "Pigou Bound". They showed that, calculating this bound is tractable and independent of network topology but the value depends on the link cost function. For example, for a network of linear cost functions (i.e. delay grows linearly with congestion) Pigou Bound is equal to 4/3. It means, distributed selfish routing cannot be worst that 4/3 of an optimal routing approach. For more reference see [15]–[17].

In this paper, we study PoA for queuing delay cost function. Surprisingly, only few results are available in the literature that derives PoA bound for this class of cost function. First, the seminal paper [18] studied the queuing delay function in the resource allocation scenario. It analyzes the global and individual optimal solution (i.e. equilibrium) for network of two nodes and parallel links with FCFS scheduling discipline and general service time distribution. However, PoA was not investigated there.[19] showed that PoA in a general network topology is unbounded by giving an example. Later, [7] proved that PoA is independent of the network topology with strong assumption that maximum amount of demand needs to be always less than the smallest link capacity in a network topology. [20] extended the results of [18] to derive PoA for the specific case of M/M/1 servers. They showed that PoA in this model is bounded tightly by the number of back-end servers. Nevertheless, only single class of data was considered in their paper. Moreover, they assume FCFS scheduling approach for each server, which is not generally QoS-enabled scheduling mechanism. Later, Altman *et al.* [21] extended result of [20] into traffic with different packet sizes. They investigate PoA for a multi-server processor-sharing system with the several Poisson input streams having different packet sizes. They conclude that PoA could be infinite for multi-class data even in simple topology consists of two nodes connected with two parallel links. However, in this model the network links serve the packets based on FCFS scheduling scheme. Therefore, there is no capacity guarantee for different classes of heterogeneous data and consequently no QoS policy.

B. Packet scheduling and PoA

Packet scheduling is a reasonable tool to provide QoS guarantees for different classes of traffic in the networks. Unfortunately, only few results are available about the impact of the packet scheduling mechanism on the behavior of a selfish routing game and its stable point. Wierman *et al.* [11] studied the application of two popular scheduling algorithms, Processor Sharing(PS) and Shortest Remaining Processing Time(SPRT). Existence of the Nash equilibrium was investigated there and PoA was derived assuming a heavy-traffic demand for a network of parallel links. Nevertheless, their scheduling algorithms do not discriminate between traffic types.

The GPS [3] is a work-conserving scheduling scheme which assumes a fluid model of network data. The GPS scheme is an extension of the the PS scheduling and allows the network designer to customize the service allocation to set of different traffic classes based on their QoS requirements. This scheduler also ensures that all service classes are guaranteed to have access to a minimum configured amount of network capacity to avoid bandwidth starvation. Moreover, GPS is considered as a fair scheduling algorithm. Also, the implementation of the GPS scheduler is cost efficient on the network links. Thus, GPS is a suitable scheduling for high speed networks the core of the Internet. To the best of our knowledge, this is the first paper which studies the effect of using GPS scheduling on the efficiency of network selfish routing.

In this paper, we aim to study the inefficiency of selfish routing when each link applies the GPS scheduling. This paper extends the work of [20] as follows: i) In our model, links use class-based scheduling - GPS- to serve the packets in contrast to FCFS scheme, which is a type-agnostic approach. ii) Selfish routing in [20] was modeled as a congestion game. However, after incorporating GPS scheduling in the same model, the cost function on a link becomes dependent on the type of other players coexisting on the same link as well. In other words, our problem becomes a weighted congestion game with player-specific cost function. [22].

III. Network Model

Consider a network represented by a graph $G = (V, E)$ where V comprised of a single pair of source and sink nodes and E is set of m parallel links which connect these two nodes. Links are modeled by M/M/1-GPS queue(i.e. packets arrive at any link with Poisson process and the being served by exponential service rate). Let $e \in E$ has the capacity μ_e and we assume without loss of generality all the links are ordered by their capacity . Let K denotes set of traffic classes where $k_{max} = |K|$. These classes ship their demands from source to sink via any subset of the links. Each class k has a demand rate D_k and is assigned a GPS weight ϕ_k where $\sum_{i=1}^{k} \phi_i = 1$. Here, ϕ_k means that, if class k has a positive backlog on the link e, then it is guaranteed to receive at least ϕ_k portion of μ_e when using that link. For simplification, we assume that packets are not allowed to change their service class while being served in the network.

At the source node, each class has a dedicated dispatcher, capable of observing the traffic of outgoing links. A Strategy of the dispatcher of class k is to select a subset of links to route its packets. Let the strategy of class k be the vector $\lambda^k = (\lambda_1^k, ..., \lambda_m^k)$ where λ_e^k is the flow of class k on the link e. Therefore, $\forall k \in K$, $\sum_{e \in E} \lambda_e^k = D_k$. Furthermore, we assume for all classes that $D_k < \sum_{e \in E} \mu_e \cdot \phi_k$. This upper bound is required to ensure the stability of the system. A system is "stable" under the condition that its total demand is less than the total network capacity. This ensures, the number of packets waiting in any link does not grow to infinity. The above upper bound results in $\sum_{k \in K} D_k < \sum_{e \in E} \mu_e$ which is equal to the stability condition mentioned above. Let the multi-strategy $P = (\lambda^{\mathbf{1}}, ..., \lambda^{\mathbf{k_{max}}})$ denote the traffic assignment setting of all classes in the network. We let the routing game be represented by a triple (G, D, C), where G is a graph topology of network of parallel links. D is a vector of demand rates for different classes(players) shown by $D = (D_1, D_2, ...D_k)$ and C is the class of GPS queuing delay cost functions. In this non-cooperative game, class dispatchers compete with each other to select a subset of links to split their demands with the goal to minimize queuing delay of their packets. For each class j and multi-strategy P we define, $E_{I_k}^P = \{e \in E \mid \lambda_e^k > 0, P = (\lambda^{\mathbf{1}}, .., \lambda^{\mathbf{k}}, .., \lambda^{\mathbf{k_{max}}})\}$ and $E_{O_k}^P = E - E_{I_k}^P$ showing the subset of links being utilized and not used respectively. In the next section we show how the link cost function is derived.

A. GPS Modeling

Consider a subset of classes in K with Poisson process arrive at the link e. Note that, because the arrival is Poisson, at each epoch of time not all classes have positive backlog on e. Let $K_{(t,e)}^{>} \subset K$ shows the set of classes which have a positive backlog on the link e at time t. Then, according to GPS scheduling algorithm, class $j \in K_{(t,e)}^{>}$ will receive a service rate of $\frac{\phi_j}{\sum_{i \in K_{(t,e)}^{>}} \phi_i} \cdot \mu_e$ at time t , while no service rate will be assigned to other classes with zero backlog. It means, $\forall t$ the class j is promised to receive at least $\mu_e \phi_j$ capacity from the link e. Modeling of average queuing delay in a GPS queue is complex problem and no closed form formula has been derived so far. For purpose of this paper, we relax the GPS scheduling model by assuming: $\lambda_e^k > 0 \iff k \in K_{(t,e)}^{>}, \forall t$. Such an approach has been utilized before [23] to drive a bound on average delay of packets in a GPS server. This assumption helps us to decompose a single M/M/1-GPS queue into a system of multiple parallel M/M/1-FCFS queues as shown in Figure 1. This is shown in Figure 1. Therefore, for rest of the paper we

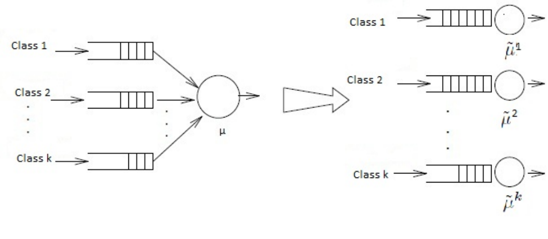

Fig. 1. Decomposition Process

use this decomposed system to formulate the packet delay cost function of the original queue GPS queue. The mean packet delay (queue and service time) of a single M/M/1 queue system with Poisson arrival rate f and exponential service rate C is already known to be $\frac{1}{C-f}$ [24]. Because the decomposition of a Poisson process remains Poisson, then each server in the decomposed system also has a Poisson arrival and can be modeled as a M/M/1 queue with the exponential service rate of $\tilde{\mu}_e^j$.

$$\tilde{\mu}_e^j = \begin{cases} 0 & e \notin E_{I_k}^P \\ \frac{\phi_k}{\sum_{j \in K_e} \phi_j} \cdot \mu_e & e \in E_{I_k}^P \end{cases} \tag{1}$$

Where $K_e = \{k \in K \mid \lambda_e^k > 0\}$. Therefore, average delay of packets of class k on the link e in a decomposed system is given below:

$$d_e^i = \frac{1}{\tilde{\mu}_e^k - \lambda_e^i} \tag{2}$$

For instance, if only user j has a positive flow on the link e, then $\tilde{\mu}_e^j = \mu_e$. However if all classes are using that link then class j only gets $\mu_e \cdot \phi_k$. The link cost and the total cost of a class dispatcher player is:

$$\forall k \in K \quad C_e^k = \lambda_e^i \cdot d_e^k, \quad C_k = \sum_{e \in E} C_e^k \tag{3}$$

We can formulate the individual cost function of players as a fractional optimization below:

$$\begin{aligned} \underset{\lambda^k}{\text{minimize}} \quad & C_e^k \\ \text{subject to} \quad & \lambda_e^k < \tilde{\mu}_e^k \qquad \forall e \in E \\ & \sum_{e \in E} \lambda_e^k \leq D_k \end{aligned} \tag{4}$$

and the aggregate cost function of a multi-strategy P denoted by C^P is: $C^P = \sum_k C_k, \lambda_k \in P$
Before analyzing PoA, we first need to characterize the existence of such an equilibrium. In case of multiple equilibria, we need to characterize the one with the worst total cost as well. Then, PoA will be analyzed by comparing the worst cost Nash equilibrium with the social optimum solution.

IV. Characterization of System Equilibria

As we discussed earlier, competition between different class of players usually leads to a stable point called Nash equilibrium where every dispatcher is satisfied with its strategy. The formal definition of a Nash equilibrium is given below:

Definition 1: Multi-strategy $\tilde{P} = (\tilde{\lambda}^1, \tilde{\lambda}^2, ..., \tilde{\lambda}^k)$ is a Nash equilibrium if for each class $k \in K$ the following condition holds:

$$\begin{aligned} C_k(\tilde{P}) &= C_k(\tilde{\lambda}^1, ., \tilde{\lambda}^{k-1}, \tilde{\lambda}^k, \tilde{\lambda}^{k+1}, ., \tilde{\lambda}^K) \\ \tilde{\lambda}^k &= \underset{\lambda^{\mathbf{k}}}{\text{argmin}}\; C_k(\tilde{\lambda}^1, ., \tilde{\lambda}^{k-1}, \lambda^k, \tilde{\lambda}^{k+1}, ., \tilde{\lambda}^{k_{max}}) \end{aligned}$$

We can give the following proposition as a consequence of the definition 1 for single class model, $|K| = 1$:

Proposition 4.1: [10] A feasible flow for a non-atomic game with monotone and continuous class of cost function is a Nash equilibrium if and only if $\forall \{e, e'\} \in E$ and single class k with $\lambda_e^k > 0$, $d_e^k \leq d_{e'}^k$.
Proof: See [10] □
Existence and uniqueness are two main issues when dealing with the Nash equilibrium point. We want to show that in this game, at least one Nash equilibrium always exists. However, this solution is not necessary unique. First, we consider a case of two nodes connected by a set of parallel links with uniform capacity. Then, we extend our results to the non-uniform capacities with some weak assumption on the demands.

A. Network of Uniform Parallel Links

1) Equilibrium Existence:

Theorem 4.2: There always exist at least one Nash equilibrium in an instance (G,D,C) when links have equal capacity

Proof: We show the existence of Nash equilibrium by construction. Assume a multi-strategy $\hat{P}$. Suppose, all class dispatchers split their demand evenly across all the links with capacity μ in the network. Thus, for each class k we have $\lambda_e^k = \frac{D_k}{m}$. Since each class is using all the links in the network, then :

$$\forall k \in K, \forall e \in E \quad \lambda_e^k > 0 \Rightarrow \tilde{\mu}_e^k = \mu \cdot \phi_k$$

Based on the proposition 4.1, $\hat{P}$ is equilibrium, since

$$\forall k \in K, \tilde{\mu}_e^k = \tilde{\mu}_{e'}^k \quad \& \quad \lambda_e^k = \lambda_{e'}^k \Rightarrow d_e^k = d_{e'}^k \tag{5}$$

□

Nevertheless, we cannot claim uniqueness of the Nash equilibrium in our problem since there exists multiple Nash equilibrium in some cases. For example, assume a simple topology, where two parallel links are shared between two classes. For two classes with configuration $D_1 = 5, D_2 = 5, \phi_1 = \phi_2 = 0.5$, settings below are the Nash equilibrium point: i)$\lambda^1 = (5, 0), \lambda^2 = (0, 5)$, ii)$\lambda^1 = (0, 5), \lambda^2 = (5, 0)$,iii)$\lambda^1 = (2.5, 2.5), \lambda^2 = (2.5, 2.5)$. For all classes, there is no benefit in changing the flow assignment for any class dispatcher. Most papers in literature assume convexity of cost function. This assumption results in uniqueness of the Nash equilibrium point which automatically makes finding PoA easier. When a game has multiple equilibria then the existence of Nash equilibrium does not suffice and the process of bounding PoA becomes tedious due to the fact that among all equilibria, the worst-case Nash equilibrium needs to be characterized.

2) Finding The Worse Equilibrium: In this section, we will characterize the Nash equilibrium which has the worst total cost. In order to find a closed form of the individual cost function, let us show a result by Haviv [20] which derives such a formula for packet queuing delay at the Nash equilibrium for FCFS scheduling and single class of traffic. (When there is only a single traffic class, FCFS scheduler operates identical to GPS).

Theorem 4.3: [20] Consider an instance of (G,D,C) where there is a single class with $D_k > 0$. There always exists an unique Nash equilibrium for this instance and the average packet delay of any packet on the links with positive flow at this Nash equilibrium is:

$$\frac{i_e}{\sum_{j=1}^{i_e} \mu_j - D_k} \tag{6}$$

$$\text{given: } i_e = \min\{i \geq 1 : \mu_{e+1} \leq \frac{\sum_{j=1}^{i} \mu_j - D_k}{i}\} \tag{7}$$

Here, i_e is the last link which has positive flow.(recall that links are sorted by their capacities)
Proof: See [20]. □
Based on equation (6), the delay in a Nash equilibrium

depends on the number of utilized links (i_e) as well as the summation of allocated service rates ($\sum_{j=1}^{i_e} \mu_j$) to the single class. This result applies to the multi-class version as well. In presence of other classes, class k receives different portion of each link $e \in E$ depending on the type of classes on that link. However, such setting is still a system of parallel M/M/1 queue only with different capacity for each class. Th main issue is whether there exists Nash equilibrium for multi-class case which is answered by Theorem 4.2. Therefore, by using Theorem 4.3, cost of class k at the Nash equilibrium $\hat{P}$ becomes:

$$C^k = D_k \cdot \frac{i_k^{\hat{P}}}{\mu^k_{(\hat{P})} - D_k} \tag{8}$$

Where $i_k^{\hat{P}} = |E_{I_k}^{\hat{P}}|$ corresponding to i_e in the formula (6) and $\mu^k_{(\hat{P})} = \sum_{e \in E_{I_k}^P} \mu_e^k$. Now, the main question arises: Given the cost at Nash equilibrium derived in (8), what assignment gives the worst Nash equilibrium when all classes are changing their strategy dynamically to minimize their individual cost function. We claim that, *the worst cost Nash equilibrium is always achieved when all classes share all links in the network.*

In the proof of existence, we implicitly showed that there always exists a Nash equilibrium P, where all classes share all the links. Now, suppose there exits another Nash equilibrium $\tilde{P}$, in which $E_{I_k}^P \neq E$. Recall that, $E_{O_k}^{\tilde{P}}$ is the subset of links on which the class k has no flow and $E_{I_k}^{\tilde{P}}$ is the subset of the links which the class k utilizes for sending its demand. For instance, we have $E_{I_k}^P = E$, $E_{O_k}^P = \emptyset$. Nash equilibrium delay cost function in (8) shows that, the delay function of a class dispatcher depends on the summation of effective capacity allocated to it (i.e. $\mu^k_{(\hat{P})}$) and number of links that have positive flow of the class k ($i_k^{\hat{P}}$). Using Theorem 4.3, cost of class k at Nash equilibrium P and $\tilde{P}$ becomes:

$$C_P^k = D_k \cdot \frac{i_k^P}{\mu^k_{(P)} - D_k}, C_{\tilde{P}}^k = D_k \cdot \frac{i_k^{\tilde{P}}}{\mu^k_{(\tilde{P})} - D_k} \tag{9}$$

Let us first state two important propositions, which are necessary to prove the main result:

Proposition 4.4: For two Nash equilibrium P and $\tilde{P}$ defined above, the following condition is sufficient to show that $C_P^k \geq C_{\tilde{P}}^k$:

$$\forall k \in K, D_k \leq \frac{i_k^P \cdot \mu^k_{(\tilde{P})} - i_k^{\tilde{P}} \cdot \mu^k_{(P)}}{(i_k^P - i_k^{\tilde{P}})} \tag{10}$$

Proof: This could be shown by minimal algebra:

$$D_k \leq \frac{i_k^P \cdot \mu^k_{(\tilde{P})} - i_k^{\tilde{P}} \cdot \mu^k_{(P)}}{(i_k^P - i_k^{\tilde{P}})} \Rightarrow$$
$$i_k^{\tilde{P}} \left(\mu^k_{(P)} - D_k\right) \leq i_k^P \cdot \left(\mu^k_{(\tilde{P})} - D_k\right) \Rightarrow$$
$$D_k \cdot \frac{i_k^P}{\mu^k_{(P)} - D_k} \geq D_k \cdot \frac{i_k^{\tilde{P}}}{\mu^k_{(\tilde{P})} - D_k} \Rightarrow$$
$$C_P^k \geq C_{\tilde{P}}^k$$

□

Since $E_{O_k}^{\tilde{P}}$ is a set of links which class k does not use at Nash equilibrium $\tilde{P}$ then no flow will switch to any link in this set. Denote $\mu_{\max} = \max_i \{\mu_i \mid i \in E_{O_k}^{\tilde{P}}\}$. The Nash equilibrium condition of the strategy $\tilde{P}$ can be stated as below:

$$\frac{i_k^{\tilde{P}}}{\mu^k_{(\tilde{P})} - D_k} \leq \frac{1}{\mu_{\max} * \phi_k} \tag{11}$$

LHS of (11) gives average delay of a packet at Nash equilibrium $\tilde{P}$ for class k and RHS is the minimum delay that a packet will experience if it switches to highest capacity in the set $E_{O_k}^{\tilde{P}}$. Equation (11) could be re-written as follows:

$$D_k \leq \mu^k_{(\tilde{P})} - i_k^{\tilde{P}} * \mu_{\max} * \phi_k \tag{12}$$

Next proposition aims to to prove this inequality:

Proposition 4.5:

$$\mu^k_{(\tilde{P})} - i_k^{\tilde{P}} * \mu_{\max} * \phi_k \leq \frac{i_k^P * \mu^k_{(\tilde{P})} - i_k^{\tilde{P}} * \mu^k_{(P)}}{(i_k^P - i_k^{\tilde{P}})} \tag{13}$$

Proof: The above inequality can be re-ordered as below:

$$-i_k^{\tilde{P}} * \mu_{\max} * \phi_k (i_k^P - i_k^{\tilde{P}}) + i_k^{\tilde{P}} \left(\mu^k_{(P)} - \mu^k_{(\tilde{P})}\right) \leq 0 \tag{14}$$

In order to prove the above inequality, we first need to calculate the term $\mu^k_{(P)} - \mu^k_{(\tilde{P})}$. Because all capacities are uniform, because of definition of P at Theorem 4.2, at Nash equilibrium P we have $E_{I_k}^P = E$ and $E_{O_k}^P = \emptyset$.

$$\mu^k_{(P)} = \sum_{e \in E_{I_k}^P} \mu_e \cdot \phi_k = \sum_{e \in E} \mu_e \cdot \phi_k =$$
$$\mu^k_{(P)} - \mu^k_{(\tilde{P})} = \sum_{e \in E_{I_k}^P} \mu_e \cdot \phi_k - \mu^k_{(\tilde{P})} =$$

We know that $E_{I_k}^P = E_{I_k}^{\tilde{P}} \cup E_{O_k}^{\tilde{P}}$. Therefore:

$$\sum_{e \in E_{O_k}^{\tilde{P}}} \mu_e \cdot \phi_k + \sum_{e \in E_{I_k}^{\tilde{P}}} \mu_e \cdot \phi_k - \sum_{e \in E_{I_k}^{\tilde{P}}} \mu_e \cdot \frac{\phi_k}{\sum_{j \in K_e} \phi_j} =$$
$$\sum_{e \in E_{O_k}^{\tilde{P}}} \mu_e \cdot (\phi_k) + \sum_{e \in E_{I_k}^{\tilde{P}}} \mu_e \cdot \left(\phi_k - \frac{\phi_k}{\sum_{j \in K_e} \phi_j}\right)$$

After replacing this into (14), we have:

$$-i_k^{\tilde{P}} \cdot \mu_{\max} \cdot \phi_k (i_k^P - i_k^{\tilde{P}}) + i_k^{\tilde{P}} \sum_{e \in E_{O_k}^{\tilde{P}}} \mu_e \cdot \phi_k$$
$$+ i_k^{\tilde{P}} \sum_{e \in E_{I_k}^{\tilde{P}}} \mu_e \cdot (\phi_k - \frac{\phi_k}{\sum\limits_{j \in K_e} \phi_j}) \leq 0$$

Note that $|E_{O_k}^{\tilde{P}}| = i_k^P - i_k^{\tilde{P}}$, thus:

$$-i_k^{\tilde{P}} \cdot \phi_k \left(|E_{O_k}^{\tilde{P}}| \cdot \mu_{\max} - \sum_{e \in E_{O_k}^{\tilde{P}}} \mu_e \cdot \phi_k \right) + i_k^{\tilde{P}} \sum_{e \in E_{I_k}^{\tilde{P}}} \mu_e \cdot \phi_k (1 - \frac{1}{\sum\limits_{j \in K_e} \phi_j}) \leq 0 \tag{15}$$

Since $\sum\limits_{j \in K_e} \phi_j \leq 1$, we conclude that the inequality (15) is always negative and consequently proposition 4.5 is always true for any class. □

After these two propositions, proof of following main theorem is straightforward.

Theorem 4.6: Consider $P, \tilde{P}$, two Nash equilibrium points in the network of parallel links with uniform capacities. Let P be a Nash equilibrium which all classes share all links given any demand while $\tilde{P}$ could be any other arbitrary equilibrium. Then we have:

$$C^{\tilde{P}} \leq C^P$$

Proof: From (10) and proposition 4.5 we conclude the inequality (10) which gives the sufficient condition to show that $C_P^k \geq C_{\tilde{P}}^k$ is true . Therefore, any improvement path from Nash equilibrium P to Nash equilibrium $\tilde{P}$ will decrease the cost of all classes which consequently reduces the total cost. □

B. Network of Non-Uniform Parallel Links

We have already proved that Nash equilibrium in which all classes share all the links has the worst cost among all other Nash equilibrium for the case of uniform links. This result could also be extended into the case when links are not uniform. Existence of such Nash equilibrium in a non-uniform case depends on the class demands. The following result establishes a demand condition under which the Nash equilibrium that all classes share all the links always exists:

Theorem 4.7: In an instance of (G, D, C), there always exists a Nash equilibrium P, where all classes share all the links if and only if $\forall k \in K$:

$$D_k \geq \sum_{e=1}^{m-1} \mu_e \phi_k - (m-1)\mu_m \phi_k \tag{16}$$

Proof: Let us first answer the question: For classless traffic what is the minimum demand such that the first m links are used at an equilibrium? Denote this demand by D_{min}. Since μ_e is decreasing in m, according to definition (7) the minimum demand is:

$$D_{min} = \min\{R \mid \mu_m \geq \frac{\sum_{e=1}^{m-1} \mu_e - R}{m}\}$$

where R is the total demand of the single class. This gives $D_{min} = \sum_{e=1}^{m-1} \mu_e - m \cdot \mu_m$. This definition extends to the multi-class demand as well. Consider a multi-strategy $\hat{P}$ in which:

$$\forall k \in K, D_k = \sum_{e=1}^{m-1} \mu_e \phi_k - m \cdot \mu_m$$

This demand for each class induces following flow on each link at Nash equilibrium:

$$\forall e \in \{1, .., m-1\}, \quad \lambda_e^k = \phi_k (\mu_e - \mu_m)$$

and a single packet on the link μ_m. This assignment is feasible since $\forall k \in K$:

$$\sum_{e \in E} \lambda_e^k = D_k, \lambda_e^k < \mu_e \cdot \phi_k$$

Multi-strategy $\hat{P}$ is a Nash equilibrium point because for each class, delay on all links are equal:

$$\forall k \in K, \forall e \in E, \tilde{\mu}_e^k - \lambda_e^k = \mu_e \phi_k - \lambda_e^k = \mu_m \phi_k$$

This result automatically extends to the case where demand is D_k, since the number of used links are non-decreasing in demand, with increasing demand, still m links will be used with but with higher capacity. □

Notice that, in the case of uniform link capacity, we did not assume any lower bound on the demand of classes. However, in the case of non-uniform links we need to assume a lower bound on the demand of classes in order to ensure the existence of such a Nash equilibrium in the network.

Theorem 4.8: Consider a network of non-uniform parallel links where demand of each class k is lower bounded by (16). Let $P, \tilde{P}$, be two equilibria where P is a Nash equilibrium which all classes share all the links, while $\tilde{P}$ is any other arbitrary equilibrium. Then we have:

$$C^{\tilde{P}} \leq C^P$$

Proof: Under this minimum demand, we proved in the Theorem 4.7 that a Nash equilibrium P always exists. Since in proof of Theorem 4.6 we did not assume any condition on link capacities, then it generalizes to non-uniform capacity and P has the worst case cost function comparing to other equilibria. □

V. Social Optimum Computation

The optimal route assignment strategy can be achieved when a centralized dispatcher decides a flow assignment for all classes on the network with the objective to minimize the summation of cost of all classes. Such global optimization is usually called *social optimal* in the literature. In a large-scale network, achieving social optimal is a very difficult task due to the excessive number of traffic sources and exponential number of routing path. We formulate the global optimization for our problem in the mathematical program below:

$$\begin{aligned} C_{optimal} = &\min \sum_{e\in E}\sum_{k\in K_e} \frac{\lambda_e^k}{\tilde{\mu}_e^k - \lambda_e^k} \\ \text{subject to} &\sum_{e\in E} \lambda_e^k = D_k, \quad \forall k \in K \\ &\sum_{k\in K_e} \tilde{\mu}_e^k < \mu_e, \quad \forall k \in K, \forall e \in E \end{aligned} \tag{17}$$

The first inequality is to ensure all classes fulfill their demand, while second one is to endure the stability condition of each link. Unfortunately, the above mathematical program is not linear nor convex since allocated capacity to the class k on the link e, $\tilde{\mu}_e^k$, depends on type of active classes on the link e (i.e. a binary value). Generally, finding a polynomial solution to such an optimization problem is not a tractable task. Nonetheless, in order to compute PoA, we need to determine the solution of social optimum. Instead of finding an exact closed form to (17), we seek to find a suitable approximation of this original cost function. Specifically, we will derive a lower bound on social optimum.

A. Bound on Social Optimum

Let us first, give a corollary which helps us to derive a lower bound on social optimum:

Corollary 5.1 (Schwarz inequality [25]): Given two sets of positive vector $A = (a_1, a_2, ...a_n)$ and $B = (b_1, b_2, ..., b_n)$, following inequality holds :

$$\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} \leq \sum_{i=1}^n \frac{a_i}{b_i}$$

Next, we give two essential propositions in order to find the social optimum bound:

Proposition 5.2: The Queuing delay cost of a M/M/1-GPS link is lower bounded by a M/M/1-FCFS link with same arrival and service rate, i.e.:

$$\frac{\sum_k \lambda_e^k}{\mu_e - \sum_k \lambda_e^k} \leq \sum_{k\in K_e} \frac{\lambda_e^k}{\tilde{\mu}_e^k - \lambda_e^k} \tag{18}$$

Proof: Let $a_i = \lambda_e^k$ and $b_i = \tilde{\mu}_e^k - \lambda_e^k$. Then, by applying Schwarz's inequality in proposition 5.1 for each link, the above inequality follows. □

Proposition 5.3: Queuing delay cost of sending demand R on m parallel link with M/M/1-FCFS model is lower bounded by cost of shipping same amount of demand on a single M/M/1-FCFS link with capacity equal to summation of all m links capacity. i.e.:

$$\frac{R}{\sum_e \mu_e - R} \leq \sum_e \frac{\lambda_e}{\mu_e - \lambda_e}$$

where $\sum_{e\in E} \lambda_e = R$

Proof: This inequality holds by letting $a_i = \lambda_e$ and $b_i = \mu_e - \lambda_e$ in proposition 5.1 □

Corollary 5.4: The social cost has following lower bound:

$$\frac{\sum_{k\in K_e} D_k}{\sum_{e\in E} \mu_e - \sum_{k\in K_e} D_k} \leq C_{optimal}$$

Proof: Given the social optimum cost, using proposition 5.2 for each link $e \in E$ we have:

$$\begin{aligned} \frac{\sum_k \lambda_e^k}{\mu_e - \sum_k \lambda_e^k} &\leq \sum_k \frac{\lambda_e^k}{\tilde{\mu}_e^k - \lambda_e^k} \Rightarrow \\ \sum_e \frac{\sum_k \lambda_e^k}{\mu_e - \sum_k \lambda_e^k} &\leq \sum_e \sum_k \frac{\lambda_e^k}{\tilde{\mu}_e^k - \lambda_e^k} \end{aligned} \tag{19}$$

On the other hand using proposition 5.3 we find :

$$\frac{\sum_e \sum_k \lambda_e^k}{\sum_e \mu_e - \sum_e \sum_k \lambda_e^k} \leq \sum_e \frac{\sum_k \lambda_e^k}{\mu_e - \sum_k \lambda_e^k} \tag{20}$$

From (19) and (20) the following can be obtained:

$$\frac{\sum_e \sum_k \lambda_e^k}{\sum_e \mu_e - \sum_e \sum_k \lambda_e^k} \leq \sum_{e\in E} \sum_{k\in K_e} \frac{\lambda_e^k}{\tilde{\mu}_e^k - \lambda_e^k}$$

Since $\sum_k D_k = \sum_e \sum_k \lambda_e^k$ then we conclude that:

$$\frac{\sum_{k\in K_e} D_k}{\sum_{e\in E} \mu_e - \sum_{k\in K_e} D_k} \leq C_{optimal}$$

□

In next section we will utilize this bound to calculate an upper bound on the Price Of Anarchy.

VI. Efficiency of Nash Equilibrium

Generally, solution given by selfish routing approach suffers performance degradation in contrast to social optimum solution. It is due to the fact that selfish routing does not take into account negative externality induced by a players on others. Using the results from last two sections we will give an upper bound on PoA of our model. First, let us give a formal definition for PoA:

Definition 2: The price of anarchy of an instance (G, D, C) is

$$PoA = \frac{\boldsymbol{C}(\lambda)}{\boldsymbol{C}(\lambda^*)}$$

Where λ is worst-cost Nash Nash equilibrium multi-strategy and λ^* is an optimal solution. Haviv [20], presented PoA in our model when there is only a single class of traffic.

Theorem 6.1: [20] In every multi-server exponential service station, the price of anarchy is at most the number of servers used in the socially optimal outcome

Proof: See [20] □

We next compare their results with our model for the case of multiple traffic classes. Using results from Theorem 4.6 and proposition 5.2 we have an upper-bound on PoA as follows:

Theorem 6.2: In the network of two nodes and m parallel links modeled by M/M/1-GPS queue where demand of each class is lower bounded by inequality shown in (16), the PoA is upper bounded by $\alpha \cdot m$ where:

$$\alpha = \frac{\sum\limits_{j\in K} \frac{T_j}{1-T_j}}{\frac{\sum_{j\in K} T_j \cdot \phi_j}{1-\sum_{j\in K} T_j \cdot \phi_j}}$$

Proof: Let P indicate the worst Nash equilibrium point. We have:

$$PoA \leq \frac{C_{WorstEquilibrium}}{\text{Bound of } C_{optimal}} = \tag{21}$$

$$\frac{\sum\limits_{j\in K} C_P^j}{\frac{\sum_{k\in K} D_k}{\sum_{e\in E}\mu_e - \sum_{k\in K} D_k}} = \frac{\sum\limits_{j\in K} D_j \cdot \frac{i_j}{\mu^j_{(P)} - D_j}}{\frac{\sum_{k\in K} D_k}{\sum_{e\in E}\mu_e - \sum_{k\in K} D_k}}$$

Since the worst Nash equilibrium occurs at the case when all classes share all the links, then $\forall k \in K, i^k = m$.

Also, for each class k we consider a normalized demand $T_k \in (0,1)$ defined as $T_k = \frac{D_k}{\sum_{e\in E}\mu_e \cdot \phi_k}$. Since $\sum_{e\in E}\mu_e \cdot \phi_k$ is the maximum demand which a class k can ship into the network, the value T_k gives the portion of maximum demand utilized by the class k.

After replacing $D_k = T_k \cdot \sum_{e\in E}\mu_e \cdot \phi_k$ into the above formula we have:

$$\frac{\sum\limits_{j\in K} \frac{m \cdot T_j \cdot \sum_{e\in E}\mu_e\phi_j}{\sum_{e\in E}\mu_e\phi_j - T_j \cdot \sum_{e\in E}\mu_e\phi_j}}{\frac{\sum_{j\in K} T_j \cdot \sum_{e\in E}\mu_e\phi_j}{\sum_{e\in E}\mu_e - \sum_{j\in K} T_j \cdot \sum_{e\in E}\mu_e\phi_j}} = m \cdot \frac{\sum\limits_{j\in K} \frac{T_j}{1-T_j}}{\frac{\sum_{j\in K} T_j \cdot \phi_j}{1-\sum_{j\in K} T_j \cdot \phi_j}} \tag{22}$$

□

Note that, this result is comparable to the results achieved by Haviv *et al.* [20] where a class-less traffic was investigated. It is interesting to note that PoA in our model is bounded by α factor of their result. The factor α could be interpreted as the impact of using GPS scheduling on the inefficiency of the selfish routing in contrast to classless model. Next section shows the effect of α on PoA. In order to compare our bound with actual PoA, next section is dedicated to some numerical results representing the accuracy of our bound with respect to the actual PoA in a simple topology.

VII. Numerical Results

In this section, we seek to evaluate the performance of our PoA bound derived in Theorem 6.2. We set up a network of two nodes, which are connected via two parallel links with capacity $\mu_1 = \mu_2 = 5$. In a non-cooperative game two class-dispatchers compete with each other to assign their demands to the network links in order to minimize their individual cost function shown in the program (4). Also, the optimization problem defined by (17) needs to be solved to give the social cost optimization. We used Wolfram Mathematica [26] to calculate both Nash equilibrium and social optimum. We assumed two class dispatchers having GPS weights $\phi_1 = 0.3, \phi_2 = 0.7$ respectively. Our program calculates the PoA bound and actual PoA for different normalized demands shown by T_1 and T_2 and $T_i \in (0,1)$. PoA bound is directly calculated using the results of Theorem 6.2. For the actual PoA, since computing the social optimal cost is not possible using an off-the-shelf polynomial algorithm, we resort to an enumeration approach : For each set of normalized demand (T_1, T_2), the social optimum cost for all permutation of class-link assignments will be calculated separately using Mathematica Optimization function (NMinimize) and an assignment with the lowest cost was selected as the social optimum point. Calculating the worst Nash equilibrium is trivial since it is always the case when all classes share all the links. Figure 2 depicts the value of α based on various demands for the case of two nodes, two links and two classes. Note that, two axis T_1 and T_2 represent the normalized demand of both classes. There are some observations based on

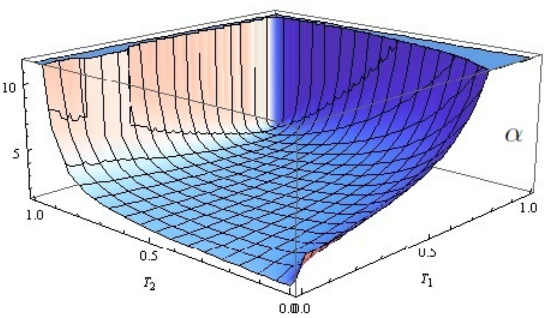

Fig. 2. Alpha Calculation

this figure: First, when value of T_1 and T_2 both increase to one (i.e. network becomes congested), then value of α will decrease. It is so since, surge in demand of all classes automatically causes the congestion on all the links in the network. As a result, in this regime, cost of social optimum solution and Nash equilibrium point move towards each other and eventually for enough large demands PoA becomes one. In other words, in a congested network, it is always recommended to implement the distributed routing protocol. Second, given $\phi_2 \geq \phi_1$, PoA has a slower growth rate when class 2 increases its demand compared to class 1. This suggests that, by assigning bigger weight to classes with higher demand we could still keep PoA small. Network designers, could utilize this guideline when they are assigning different GPS weight to classes of traffic. Likewise, given $\phi_1 < \phi_2$ if $T_1 > T_2$, PoA increase exponentially. This might indicate that if users have higher priority, they should have also high demand. Otherwise, they harm the overall benefit of the network. Figure 3 depicts both the PoA bound and the actual PoA:

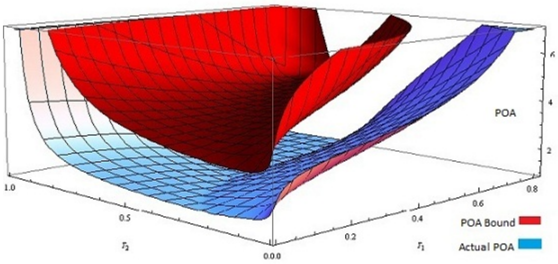

Fig. 3. PoA Performance

VIII. Conclusion

We have studied the inefficiency of selfish routing in a network of parallel links with the multi-class traffic. We focused on a network with GPS scheduler with guaranteed capacity reservation. We showed that, the performance of selfish routing is highly dependent on demand of classes and the QoS parameters of GPS policy. We proved that when the back-end scheduler is GPS, PoA in a selfish routing is worse than the model of of routing game with a single traffic and with FCFS scheduler by a factor of α which could be arbitrary large. Furthermore, by analyzing this factor in a network of parallel links, we showed that the inefficiency of class-based selfish routing is dependent on the characteristics of the classes and their demands shown by α as well as the number of links in the topology. We conclude that the PoA can be greatly improved as long as classes with higher GPS weight also have higher traffic rate and classes with small GPS weight send lower demands . These results provide interesting insight into the behavior of the heterogeneous system in the network model with different class of data and may have implications in the debate on net neutrality.

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