Cooperative Online Native Advertisement: a Game Theoretical Scheme Leveraging on Popularity Dynamics

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***Abstract*—We propose a model for native online advertising, a recent technique able to safeguard user's online experience and yet greatly accelerate the virality of a content. As of now, the cost of native advertisement represents a high entrance barrier for potential customers of advertisement hosts. In this paper, we argue that cooperative schemes can lower this entrance barrier. In fact, when different content providers (CPs) advertise their products on the same web-page, virality of the web-page itself can boost the number of potential customers, causing a non linear dynamics in the revenue of CPs. By utilizing a cooperative game theoretical framework, we derive a sufficient condition, depending on system's parameters, under which it is profitable for different CPs to advertise their products together. Under a mild assumption, the condition is also shown necessary.**

***Index Terms*—native advertisement, social media, virality, cooperative games.**

I. Introduction

Online advertisement has become a strategic source of business led by corporate enterprises such as Google, Facebook and Twitter. Indeed, the retailer industry is driven more and more by online advertisement [7]. Several examples of online advertisement exist. For instance, based on the celebrated search engine, Google AdWords provides online advertisement. Using AdWords tools, a service provider or a retailer can perform advertisement towards a target audience and customize the message and the reach of the campaign both from the geographical and social standpoint. Using YouTube video promotion, online video advertisement can be pushed towards interested users by means of embedded videos or recommended promoted advertisements. Recently, social networks followed this trend too. E.g., Facebook is letting advertisements appear directly onto users' Facebook timeline.

However, all the above examples are to be considered, to some extent, "invasive" ads, since the target audience of such ads may perceive them as annoying or, even worse, as violating a private area, e.g., the timeline of a Facebook user.

To this respect, *native advertising* is the novel paradigm expected to perform a major shift in online advertising. Quoting [5], *"Native ads are ads in a format that is native to the platform on which they are run, bought or sold. Native advertising is the activity of producing, buying and selling native ads."* In practice, native advertisements become hardly distinguishable from ordinary contents that the user watches on the host pages. But, at the right moment, a link to a certain item to be purchased appears without much nuisance to the user's online experience. Fig.1 reports on two case studies of native advertising as presented on a specific host portal.

Ultimately, the aim of native advertising is to help contents, and thus the attached advertisement and products, to go *viral*. Actually, recent statistics [3] suggest that users are indeed more willing to share native ads on social networks rather than standard display ads.

However, despite native advertisement appears a promising channel, in its current format it is bound to be confined to enterprise-level digital advertising. In fact, these ads require a strict cooperation with the publisher. The costs to be negotiated with marketing platforms such as YouTube, Facebook, Twitter or with publishers such as Forbes BrandVoice, do not appear sustainable neither for small to medium size brands nor for small retailers.

The model developed in this work introduces a new solution concept for online advertisement and proves that it is possible to make native advertisement affordable to small and medium online retailers. Currently, native advertising is performed in an outright competitive fashion, in which each content provider (CP) advertises its own product on different web-pages. We argue that it would be sometimes profitable for CPs to *jointly advertise* their products. In particular, by utilizing game theoretical tools, we develop a cooperative model based on the assumption that the publisher accepts *coalitions* of advertisers jointly taking part to a native advertisement campaign. This would be a scheme similar to those used in aggregation web-pages for online retailing. However, the novelty here comes from the dynamics of online contents: when a native advertisement succeeds, indexing of pages increases the set of target potential customers of a certain item at a dramatic speed.

Main contribution. In this paper we identify conditions under which cooperative native advertisement schemes become beneficial for CPs. To the best of the authors' knowledge, this is also the first work deriving models for online advertisement by means of cooperative games. In particular, we elaborate models leveraging on the major enlargement of the customer base induced by viral marketing. This key feature is the core advantage of online advertisement over social media. We

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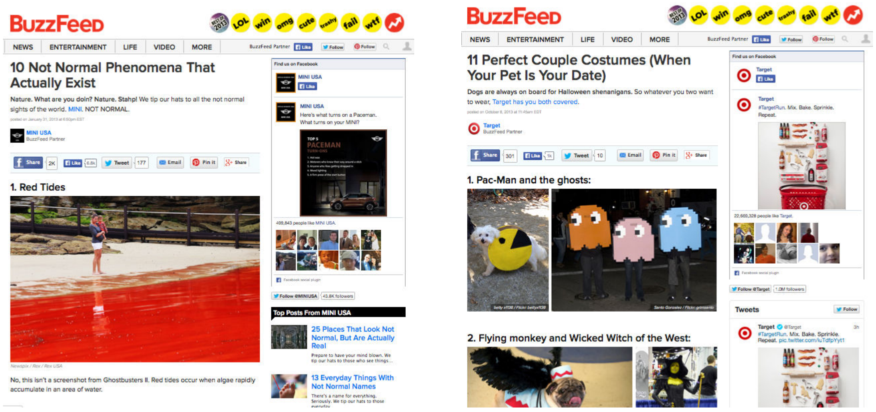

(a) (b)

Figure 1: Two sample native advertisements. The commercial purpose of the message is not apparent at first sight. But, it is part of the displayed content. Fig.1a is a promotion campaign for a car. The main page aligns to the promotion campaign showed in the right banner in a subtle manner: the "not normal" motto of the advertiser rebounces in a seamless fashion to the attention of the page visitor. In Fig.1b we observe a custom page that was developed for a retailer of pet accessories; nevertheless, it is not so obvious that products for purchase are on display. It becomes so after following a link that leads to an ordinary purchase web-page.

show that, by aggregation of several appealing contents, online native advertisers, or CPs, can attain much larger hit rates to their products than trying to promote them alone. In fact, by promoting several products on the same web-page, such a web-page may go viral sooner. Using the theory of cooperative games with transferable utility, we show that, depending on the parameters of the system, a coalition of CPs will be able to form and be stable, i.e., every CP taking part would not be better off by advertising on separate web-pages.

II. System Model

We consider M different content providers (CPs) that promote online their respective contents, or products, via native advertisement. We call $\mathcal{M} = \{1, 2, \dots, M\}$ the set of CPs. When a native advertisement web-page has just been posted online, the content of each CP is initially visible to a "small" set $\mathcal{N}$ of N potential customers. Furthermore, we assume that a user will eventually purchase only one item of the products advertised by the different CPs. The case of multiple simultaneous purchases is out of the scope of this paper.

Purchase dynamics: We model the evolution of the number of contents purchased over time by using a Poisson spreading process, already adopted, e.g., in [1], [2]. We assume that a CP's product is bought by online customers according to a Poisson process of parameter λ, i.e., the content of any CP is bought every λ^{-1} unit of time. Hence, λ is a measure of the popularity of a content. We further assume that all contents have same cost c. The case of heterogeneous costs and popularity will be part of future works.

Exposure boost: As introduced before, native advertisement is a powerful and recent method that allows the advertisement of a product to go viral, i.e. to receive a sudden boost in its exposure to online users. The exposure boost of native advertisements may depend on many factors. Among them, media buzz is a prominent example. Indeed it often occurs that, whenever a web-page is "popular enough", it receives the attention of the media that expose it to people that have never heard of that specific web-page before. Thus, the content advertised on that web-page receives a sudden exposure boost. Diffusion over multiple social networks may follow up, and bolster the virality of the advertisement. In order to model this phenomenon, we measure the popularity of an advertising web-page as the number of purchases performed via the web-page itself. Then we assume that when the number of purchases exceeds a threshold Ψ ($< N$), the web-page receives an exposure boost and becomes visible to a larger set of potential consumers $\mathcal{N} \cup \delta\mathcal{N}$ ($\delta N = |\delta\mathcal{N}|$).

Time horizon: We assume that the native advertisement campaign has a finite duration T. The CPs aim at maximizing their profit at the end of the time interval $[0, T]$.

Joint advertising: In our scheme, we assume that any group $\mathcal{S} \subseteq \mathcal{M}$ of CPs has the possibility to advertise its products leveraging on a common aggregation page. In this paper we intend to study under which conditions CPs are willing to take part to the same aggregation page. Those who appear on the same aggregation page, in practice, form a coalition because they agree in being viewed together, thus sharing a base of common customers. Note that the competition aspect is still there, since different CPs *compete* over the same consumer base, and each consumer will eventually purchase only one of the M contents. However, they *cooperate* in the hope that the popularity of the common web-page on which they advertise goes viral and is accessible by a larger number of potential customers for a longer period of time.

By utilizing some useful concepts and tools from cooperative game theory, we study the sustainability and profitability of cooperative native advertising for a group of online CPs. Depending on the system parameters, several retailers may or may not form a coalition and advertise or not their products on

Table I: Main notation used throughout the paper

Symbol	*Meaning*
$\mathcal{M}$	Set of content providers (CPs), $M = \|\mathcal{M}\|$.
$\mathcal{S}$	Coalition of CPs, $\mathcal{S} \subseteq \mathcal{M}$, $S = \|\mathcal{S}\|$.
$\mathcal{N}$	Initial set of potential customers, $N = \|\mathcal{N}\|$.
$\delta\mathcal{N}$	Set of novel potential customers after the advertising web-page goes viral, $\delta N = \|\delta\mathcal{N}\|$.
Ψ	Threshold that the purchase count of a web-page must exceed to go viral.
λ	Number of purchases per unit of time for a content.
$x^c(t)$ $(\delta x^c(t))$	Number of customers belonging to $\mathcal{N}$ ($\delta\mathcal{N}$) having purchased the content of one CP before time t in the *c*ompetitive model.
$X^{\mathcal{S}}(t)$ $(\delta X^{\mathcal{S}}(t))$	Number of customers belonging to $\mathcal{N}$ ($\delta\mathcal{N}$) having purchased contents of the CP belonging to the coalition $\mathcal{S}$ before time t.
T	Time horizon.
$\underline{t}^S$	Time at which a coalition $\mathcal{S}$ of S CPs attains the threshold Ψ.
c	Unit cost of the product advertised by a CP.
r^c	Profit for a CP in the competitive model.
$R^{\mathcal{S},\mathcal{P}}$	Profit for coalition $\mathcal{S}$ when the CPs $\overline{\mathcal{S}}$ cooperate as in $\mathcal{P}$.
$v(S)$	Value of a coalition $\mathcal{S}$ of CPs with S members.

the same web-page. All the proofs of our results are deferred to the Appendix.

III. Outright competition among content providers (CPs)

We start analyzing the model in which CPs do *not* cooperate, i.e., they do not perform advertisement of the their products jointly on the host portal, e.g., through a common advertisement web-page.

We remark that this represents the current situation with native advertising: companies are assigned dedicated advertisement web-pages and do not share them with any other advertiser. As explained before, we consider the Poisson evolution process of the number of purchases, under the assumption that online customers would purchase only one item among the contents available on the host portal.

Let us start by providing an expression for $x^c(t)$, which is the number of users among $\mathcal{N}$ who purchased a content of one CP before time t under the *c*ompetitive assumption. The fluid approximation of $x^c(t)$ satisfies the following differential equation [1]:

$$\frac{d}{dt}x^c(t) = \lambda\big(N - Mx^c(t)\big), \quad t \in [0;T], \ i \in \mathcal{M} \tag{1}$$

whose unique solution is

$$x^c(t) = \frac{N}{M}\left(1 - e^{-M\lambda t}\right), \quad t \in [0;T] \tag{2}$$

Notice that, by the homogeneous assumption, the expression of $x^c(t)$ does not depend on the specific CP.

The effect of the online advertisement is represented by a sharp increase of visibility, i.e., a sharp enlargement of the customer base: when x^c exceeds some purchase threshold Ψ, content i becomes viral and so it becomes visible also to the additional set of customers δN.

The time instant $\underline{t}^c$ at which the view count $x^c(t)$ exceeds the threshold is

$$\underline{t}^c = -\frac{1}{M\lambda}\log\left(1 - \frac{\Psi M}{N}\right), \quad \text{s.t. } x^c(\underline{t}^c) = \Psi$$

As long as $x^c(t)$ does not attain the threshold $\Psi < N$, contents do not go viral and do not reach out to the additional consumer base $\delta\mathcal{N}$. Hence, $\delta x^c(t) = 0$ for $0 \le t \le \underline{t}^c$. Note that $\underline{t}^c > T$ whenever $x^c(T) < \Psi$.

Let us find an expression for $\delta x^c(t)$ when the content i has become viral. The definition of δx^c is the analogous of x^c for the consumer base $\delta\mathcal{N}$. Since contents become viral at time $\underline{t}^c$, the purchase count δx^c origins at time $\underline{t}^c$ as well. Hence, in a similar fashion to expression (2), we can write

$$\delta x^c(t) = \frac{\delta N}{M}\left(1 - e^{-M\lambda(t-\underline{t}^c)}\right) = \frac{\delta N}{M}\left(1 - \frac{N}{N - M\Psi}e^{-M\lambda t}\right). \tag{3}$$

The final revenue r^c for each CP equals the purchase count at time $t = T$ multiplied by the cost of one advertised item, i.e.,

$$r^c = c\big(x^c(T) + \delta x^c(T)\big).$$

IV. Cooperation among content providers

After dealing with the situation in which CPs are in outright competition among each other in Section III, we are finally ready to analyze the model in which any group $\mathcal{S} \subseteq \mathcal{M}$ of CPs can join forces. I.e., we assume that the native advertising host allows several CPs to advertise their products on the same native advertising web-page. In this case, we say that the CPs in $\mathcal{S}$ cooperate, or equivalently that coalition $\mathcal{S}$ forms. Specifically, in this paper we intend to study when it is beneficial for all CPs to cooperate with each other and share the resulting revenue, and form the so-called grand coalition $\mathcal{M}$.

We call $X^{\mathcal{S}}(t)$ the number of users in $\mathcal{N}$ that purchased a content of any CP belonging to $\mathcal{S}$ before time t, when the CPs in $\mathcal{S}$ form a coalition, and it writes

$$X^{\mathcal{S}}(t) = Sx^c(t) = \frac{S}{M}N\left(1 - e^{-M\lambda t}\right) \quad t \in [0;T] \tag{4}$$

Note that the CPs of the anticoalition $\overline{\mathcal{S}} = \mathcal{M}\backslash\mathcal{S}$ are allowed to organize themselves in subcoalitions. Nevertheless, $X^{\mathcal{S}}$ does not depend on the behavior of $\overline{\mathcal{S}}$.

As before, we assume that when the number of purchases performed on the native advertising web-page exceeds the threshold Ψ, the web-page goes viral and receives an exposure boost. When it does, it becomes accessible to the additional users $\delta\mathcal{N}$. We recall that our aim is to analyze under which conditions joint native advertising proves beneficial for all the members of the coalition.

We let $\underline{t}^S$ be the time when the purchase count $X^{\mathcal{S}}$ of coalition $\mathcal{S}$ attains the threshold Ψ:

$$\underline{t}^S = -\frac{1}{M\lambda}\ln\left(1 - \frac{\Psi M}{NS}\right).$$

Again, $\underline{t}^S > T$ when $X^{\mathcal{S}}(T) < \Psi$.

Let $\mathcal{P}$ be a partition of $\overline{\mathcal{S}}$, enlisting the subcoalitions that form inside the anticoalition $\overline{\mathcal{S}}$. We call $\delta X^{\mathcal{S},\mathcal{P}}(t)$ the number of users in $\delta\mathcal{N}$ that purchase a content of a CP in $\mathcal{S}$ before time t, under the condition that coalition $\mathcal{S}$ forms and the anticoalition $\overline{\mathcal{S}}$ is fragmented as in $\mathcal{P}$. We observe that indeed $\delta X^{\mathcal{S},\mathcal{P}}(t)$ does depend on how fragmented the anticoalition $\overline{\mathcal{S}}$

is. More precisely, $\mathcal{P}$ determines at what time the different coalitions in $\overline{\mathcal{S}}$ go viral. Then, the revenue that each CP belonging to coalition $\mathcal{S}$ obtains at the end of the time horizon, i.e., $t = T$, is

$$r^{\mathcal{S},\mathcal{P}} = \frac{c}{S}\big(X^{\mathcal{S}}(T) + \delta X^{\mathcal{S},\mathcal{P}}(T)\big)$$

where $S = |\mathcal{S}|$. The revenue of coalition $\mathcal{S}$ is the sum of the revenues of its members:

$$R^{\mathcal{S},\mathcal{P}} = S r^{\mathcal{S},\mathcal{P}}. \tag{5}$$

A. Cooperative game

In this paper we study when CPs can effectively leverage on cooperation in order to boost popularity. More specifically, we wish to find under which conditions CPs are willing to advertise their products on common aggregation web-pages by redistributing among themselves their total revenue $R^{\mathcal{M}} = M r^{\mathcal{M}}$ in a "satisfactory" manner for each of them. In order to tackle this problem, and define exactly when a revenue reallocation can be considered "satisfactory" for all CPs, we utilize the framework of cooperative game theory (CGT) with transferable utility (TU), and in particular the concept of *Core* of the game, that we will define later.

CGT describes the situation in which a group of players $\mathcal{M}$ can stipulate binding agreements among each other and form coalitions. When coalition $\mathcal{S} \subseteq \mathcal{M}$ forms, the real value $v(\mathcal{S})$ is assigned to $\mathcal{S}$. Under the TU assumption, $v(\mathcal{S})$ can be allocated in any manner among the members of the coalition $\mathcal{S}$. In our case, this assumption is sensible, since $v(\mathcal{S})$ will be later defined as the profit of coalition $\mathcal{S}$ at time T, which is a transferable quantity by definition.

CGT mostly studies how to reallocate the value $v(\mathcal{M})$ achieved by the grand coalition $\mathcal{M}$ among its members, assuming that $\mathcal{M}$ forms. The Core is one of the most prominent allocation concepts in CGT, and well expresses the notion of "satisfaction" for the member of the grand coalition.

Definition 1. *The* Core[1] *is the set of allocations $y \in \mathbb{R}^M$ s.t.*

$$\sum_{j\in\mathcal{M}} y_j = v(\mathcal{M}) \tag{6}$$

$$\sum_{j\in\mathcal{S}} y_j > v(\mathcal{S}), \qquad \forall\, \mathcal{S} \subset \mathcal{M}. \tag{7}$$

Equation (6) states that an allocation x in the Core of the game (v, M) is actually a reallocation of $v(\mathcal{M})$ among the CPs, while expression (7) claims that there is no way for any coalition $\mathcal{S}$ of CPs to withdraw from the grand coalition, share among themselves the resulting revenue $v(\mathcal{S})$, and ensure to all of its members a better (or equivalent) allocation. In this case, we say that the grand coalition is stable.

In our model, we will see that an allocation belongs to the Core whenever for all CPs it is profitable to cooperate by advertising their products on a common aggregation web-page, and no subcoalition can achieve a better revenue by advertising their products on a different web-page.

The value $v(\mathcal{S})$ can be defined in several ways (see [6] for a thorough survey), but one of its most widely accepted interpretations is the revenue that $\mathcal{S}$ can ensure for itself when the anticoalition adopts an antagonistic behavior. Such a worst-case scenario approach was first introduced in [8]. Hence, in our model, the value of coalition $\mathcal{S}$ is computed as the maximum revenue that $\mathcal{S}$ can achieve when the cooperation strategy of the anticoalition $\overline{\mathcal{S}}$ is the least profitable for $\mathcal{S}$. In our case, the value of a coalition only depends on its cardinality. Hence, for simplicity we will simply denote the value of coalition $\mathcal{S}$ as $v(S)$, where $S = |\mathcal{S}|$, and the value of the grand coalition as $v(M)$. Hence we can write

$$v(S) = \min_{\mathcal{P}} R^{\mathcal{S},\mathcal{P}}, \qquad \forall\, \mathcal{S} \subseteq \mathcal{M}, \tag{8}$$

where $\mathcal{P}$ is a generic partition of $\overline{\mathcal{S}}$. The following Lemma tells us that, at it is intuitive, the less profitable scenario for coalition $\mathcal{S}$ is when all members in $\overline{\mathcal{S}}$ cooperate with each other.

Lemma 1. *The minimum in (8) is achieved when all CPs of anticoalition $\overline{\mathcal{S}}$ cooperate with each other, i.e.*

$$v(S) = R^{\mathcal{S},\{\overline{\mathcal{S}}\}} = c\left(X^{\mathcal{S}}(T) + \delta X^{\mathcal{S},\{\overline{\mathcal{S}}\}}(T)\right). \tag{9}$$

The value $v(\mathcal{M})$ of the grand coalition, i.e., the revenue that CPs will eventually share among themselves, is then simply the sum of the revenues of each CP when they all cooperate:

$$v(\mathcal{M}) = R^{\mathcal{M}}.$$

Hereafter we will simply denote $\delta X^{\mathcal{S},\{\overline{\mathcal{S}}\}}(T)$ as $\delta X^{\mathcal{S}}(T)$, and $R^{\mathcal{S},\{\overline{\mathcal{S}}\}}$ as $R^{\mathcal{S}}$.

B. Computation of coalition values

Lemma 1 prescribes how to compute the value $v(\mathcal{S})$ of each coalition $\mathcal{S} \subseteq \mathcal{M}$. To this aim, we still have to compute the dynamics $\delta X^{\mathcal{S}}$. In fact, we already know the expression of $X^{\mathcal{S}}$ from (4).

Let us compute the purchase count achieved by the grand coalition $\mathcal{M}$:

$$\delta X^{\mathcal{M}}(t) = \delta N\left(1 - e^{-M\lambda t}\frac{N}{N-\Psi}\right). \tag{10}$$

Then we consider the generic coalition $\mathcal{S}$, and we analyze the situation in which $\underline{t}^{S} < \underline{t}^{\overline{S}}$, i.e. $\mathcal{S}$ attains the threshold Ψ before the anticoalition $\overline{\mathcal{S}}$. This occurs when $S > M/2$. Then $\delta X^{\mathcal{S}}(t) = 0$ for all $t \in [0; \underline{t}^{S})$, and

$$\begin{aligned}
\delta X^{\mathcal{S}}(t) =& \quad \delta N\left(1 - e^{-S\lambda(t-\underline{t}^{S})}\right), \quad t \in [\underline{t}^{S}; \underline{t}^{\overline{S}}] \\
\delta X^{\mathcal{S}}(t) =& \quad \delta X^{\mathcal{S}}(\underline{t}^{\overline{S}}) + \big(\delta N - \delta X^{\mathcal{S}}(\underline{t}^{\overline{S}})\big)\tfrac{S}{M} \times \\
& \times \left(1 - e^{-M\lambda(t-\underline{t}^{\overline{S}})}\right), \quad t \in (\underline{t}^{\overline{S}}; T]
\end{aligned} \tag{11}$$

If $\underline{t}^{S} < T < \underline{t}^{\overline{S}}$, i.e. the contents of the CPs belonging to $\overline{\mathcal{S}}$ are never exposed to the users in $\delta\mathcal{N}$, then

$$\delta X^{\mathcal{S}}(T) = \delta N\left(1 - e^{-S\lambda T}\Big(1 - \frac{\Psi M}{NS}\Big)^{-\frac{S}{M}}\right). \tag{12}$$

[1]Our notion of Core is typically referred to as *strict Core*; the standard notion of Core implies the $\geq$ sign in expression (7)

If $S = M/2$, then the coalition $\mathcal{S}$ and the anticoalition go viral at the same instant, and

$$\delta X^{\mathcal{S}}(t) = \frac{\delta N}{2}\left(1 - \frac{N}{N-2\Psi}e^{-M\lambda t}\right).$$

When $\underline{t}^{\overline{\mathcal{S}}} < \underline{t}^{\mathcal{S}}$, i.e., $S < M/2$ then $\delta X^{\mathcal{S}}(t) = 0$ for $t \le \underline{t}^{\mathcal{S}}$, and

$$\delta X^{\mathcal{S}}(t) = \left(\delta N - \delta X^{\overline{\mathcal{S}}}(\underline{t}^{\mathcal{S}})\right)\frac{S}{M}\left(1 - e^{-M\lambda(t-\underline{t}^{\mathcal{S}})}\right)$$

for all $t \in [\underline{t}^{\mathcal{S}}, T]$, where $\delta X^{\overline{\mathcal{S}}}(\underline{t}^{\mathcal{S}}) = \delta N(1 - e^{-\overline{S}\lambda(\underline{t}^{S} - \underline{t}^{\overline{S}})})$. The value of each coalition is finally computed as in (9).

V. Core: when cooperation is profitable

In this section we present our main result on the existence of the Core. Before that, we can already observe two simple facts.

Remark 1 (No virality)**.** *If the grand coalition $\mathcal{M}$ does not attain the threshold Ψ, i.e., $\underline{t}^{M} \ge T$ (thus $\underline{t}^{S} > T$ for any $\mathcal{S}$) then $v(S) = Sv(1)$ for all $\mathcal{S} \subseteq \mathcal{M}$. This means that if we do not consider the virality assumption in our model, then it is indifferent for the CPs to cooperate or not, since the resulting revenue would be exactly the same in the two cases. Hence,* the Core is empty.

Remark 2 ($\mathcal{M}$ the only viral)**.** If the grand coalition is the only coalition able to attain the threshold Ψ, *i.e., $\underline{t}^{M} < T$ and $\underline{t}^{S} \ge T$ for all $\mathcal{S} \subset \mathcal{M}$, then $v(M) > Sv(1)$ and $v(S) = Sv(1)$ for all $\mathcal{S} \subset \mathcal{M}$. Then,* the Core has infinitely many elements*, of the form $y_i = v(1) + d_i$ with $d_i > 0$, $\sum_{j\in\mathcal{M}} d_j = v(M) - Mv(1)$.*

Now we are ready to present our main result. It provides a sufficient condition for the Core to be nonempty. Under such condition, uniformly distributing the total revenue $v(M)$ among the CPs is profitable for all CPs. This is because no subset of CPs would find more profitable to advertise their products on a different web-page. Moreover, such a revenue allocation is also more profitable than the situation in which no cooperation is allowed, described in Section III.

Remarkably, our condition on the nonemptiness of the Core is also necessary if *no* single CP can go viral on its own and attain the threshold Ψ.

Theorem 1. *Assume that $\mathcal{M}$ goes viral*[2]*, i.e., $\underline{t}^{\mathcal{M}} < T$. Let*

$$g(x) = x^{-1}\left[1 - e^{-x\lambda T}\left(1 - \frac{\Psi M}{Nx}\right)^{-\frac{x}{M}}\right],$$

for $1 \le x \le M$. Then,

i) If

$$g(M) \ge g(M-1) \tag{13}$$

then the Core of the cooperative game is nonempty and the allocation

$$y_i^* = r^{\mathcal{M}} = v(M)/M, \qquad i = 1, \dots, M$$

belongs to the Core;

ii) If no single CP attains the threshold Ψ on its own, i.e. $\underline{t}^{1} \ge T$, then (13) is also necessary for the Core to be nonempty.

iii) The allocation y^* is more profitable for each CP than the revenue obtained in the absence of cooperation, i.e.,

$$y_i^* \ge r^c.$$

[2]The case $\underline{t}^{\mathcal{M}} \ge T$ is covered by Remark 1

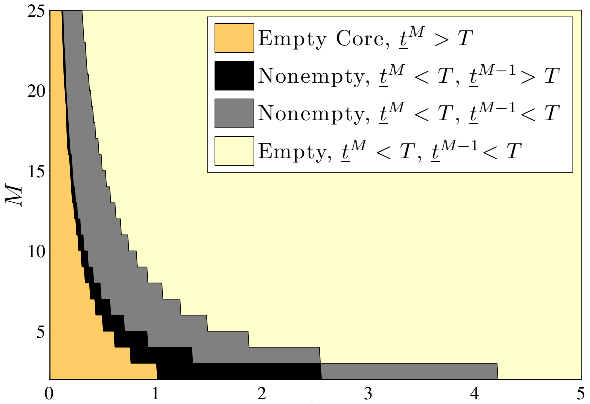

Figure 2: The dark region of the figure indicates at which (λ, M) pairs the Core is nonempty, with $N = 10^5$, $\delta N = 5N$, $T = 0.3$, $\Psi = 6 \times 10^4$. In the yellow area region, in which $\underline{t}^M > T$, no coalition ever attains the popularity threshold. Hence, the values are just linear in the size of the coalition and the Core is empty (see Remark 1). In the black region, with $\underline{t}^M < T, \underline{t}^{M-1} > T$, the grand coalition is the only coalition attaining the threshold Ψ, hence the Core is nonempty (see Remark 2). In the light grey region, with $\underline{t}^M < T, \underline{t}^{M-1} < T$, the Core still exists - not trivially - even if there are some subcoalitions capable to attain the threshold and hence go viral. Notice that in this case, condition (13) is necessary and sufficient for the Core to be nonempty, since no single CP can ever attain the threshold ($\Psi > N/2$). The yellow area accounts for those (λ, M) pairs such that the Core is empty and, still, $\underline{t}^M < T, \underline{t}^{M-1} < T$.

By the argument of symmetry, redistributing $v(M)$ equally among all CPs is the most reasonable way to allocate the revenue under the homogeneous assumption. Hence, Thm. 1 can be reinterpreted as the condition under which the uniform allocation $y_i = v(M)/M$, $i = 1, \dots, M$, also belongs to the Core, i.e. it is satisfactory for all CPs. A couple of interesting observations follow from the result in Thm. 1.

Remark 3 (No reallocation needed)**.** *Thm. 1 claims that the revenue $r^{\mathcal{M}}$, that each CP $i = 1, \dots, M$ obtains, already belongs to the Core. Hence, the revenue that each CP obtains in the first place is already satisfactory for each of them, without the need of redistributing the revenue among the CPs. In general, even if the Core is nonempty, this may not be true, and it might be necessary to resort to the transferable utility assumption in order to reallocate the overall revenue $v(M)$ in order to make the grand coalition stable.*

Remark 4 (Coalition formation)**.** *When the Core is empty, it still does not mean that no cooperation at all is possible. Indeed, we should expect that the grand coalition fragments itself into smaller coalitions of CPs, which advertise their products on different web-pages. This situation can be studied by utilizing concepts from coalition formation theory [4]. Nevertheless, this is out of the scope of this paper.*

We conclude our analysis with some numerical experiments shown in Figure 2. They outline that, once we fixed all parameters but M and λ, cooperation among CPs is guaranteed when the product λM is approximately constant: when the number of CPs M is large, then their popularity λ should be small enough, and viceversa. We also highlighted different (M, λ) areas; they correspond to different conditions that enforce a nonempty Core.

VI. Conclusions

Native advertising has been recent found effective in making a content viral. In this paper we propose a cooperative online advertisement strategy for content providers (CPs) via native advertising. Instead of simply advertising their products on separate web-pages, we study the situation in which different CPs cooperate. Hence, they can advertise their respective products on a common aggregation web-page which goes viral when it generates a sufficiently large number of purchases. We utilize the cooperative game theoretical framework, and more specifically the concept of Core, to study conditions under which cooperation is profitable for all CPs.

We observed that, without considering the virality assumption, cooperation is not profitable (see Remark 1). Conversely, we derived a sufficient condition under which splitting the total revenue uniformly among the CPs belongs to the Core (Thm. 1, *i)*). Such condition is also necessary for the Core to be nonempty under a mild assumption (Thm. 1, *ii)*). Moreover, the situation with cooperation is always at least as profitable as the situation described in Section III, in which cooperation is not allowed. This follows from the fact that, by cooperating, CPs are able to make their contents go viral sooner, whereas they might not make it by going alone.

Finally, numerical experiments suggest that we should typically expect cooperation within a small group of very popular CPs, or else within a numerous set of "less fashionable" CPs.

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Appendix

A. Proof of Lemma 1

Proof: Because of lack of space, we provide a sketch of the proof. We need to prove that $\delta X^{\mathcal{S},\mathcal{P}}(T) \geq \delta X^{\mathcal{S},\{\overline{\mathcal{S}}\}}(T)$ for any partition $\mathcal{P}$ of $\overline{\mathcal{S}}$. We also notice that $\underline{t}^{\overline{\mathcal{S}}} < \min_{P \in \mathcal{P}} \underline{t}^P$. Hence, when all CPs belonging to $\overline{\mathcal{S}}$ cooperate with each other, they all have access to the consumer base $\delta\mathcal{N}$ in advance w.r.t. the situation in which $\overline{\mathcal{S}}$ is fragmented. Hence, the probability that a user purchases a content of a CP belonging to the coalition $\mathcal{S}$ is minimized when $\overline{\mathcal{S}}$ is united. Then, the expected number of purchase count for the coalition $\mathcal{S}$ is minimized too. ∎

B. Proof of Theorem 1

Proof: Thanks to Prop. 1 (see below), we can say that the Core is nonempty if and only if $v(M)/M \geq v(S)/S$. We notice that $v(M) = \delta NMg(M) + Mx^c(T)$, and $v(S) \leq \delta NSg(S) + Sx^c(T)$ for all $S < M$, where equality occurs only when the anticoalition $\overline{\mathcal{S}}$ does not attain the threshold (compare with Eq. 12). Since the function $g(x)$ has only one local maximum in $1 \leq x \leq M$, then $g(M) \geq g(S)$ for all $S < M$ if and only if (13) is verified. Thus, if (13) holds then we can write

$$\frac{v(M)}{M} = \delta Ng(M) + x^c(T) \geq \delta Ng(S) + x^c(T) \geq \frac{v(S)}{S},$$

for $1 \leq S < M$. Hence the Core is nonempty, and *i)* is proven. Now, let us prove *ii)*. Let us assume that the Core is nonempty. Then, for Prop. 1, $v(M)/M \geq v(M-1)/(M-1)$. Since no single CP can attain the threshold, i.e. $\underline{t}^1 \geq T$, then $v(M-1)/(M-1) = \delta Ng(M-1) + x^c(T)$. Also, we know that $v(M)/M = \delta Ng(M) + x^c(T)$. Hence, (13) holds and *ii)* is proven. The claim *iii)* is proven by inspection on the expressions (3) and (10). ∎

Proposition 1. *Let (M, v) be a cooperative game with transferable utility. Assume that the coalition values v only depend the cardinality of the coalition. Then, the Core of the game is nonempty if and only if*

$$\frac{v(M)}{M} > \frac{v(S)}{S} \qquad \forall \mathcal{S} \subset \mathcal{M}. \tag{14}$$

Proof: Assume that (14) holds. Let $y_i = v(M)/M$ be the allocation to player i. Then, $\sum_{i \in \mathcal{M}} y_i = v(M)$ and $\sum_{i \in \mathcal{S}} y_i = S\frac{v(M)}{M} > v(S)$. Hence, y belongs to the Core of the game. Conversely, assume that the Core is nonempty, and y' is in the Core. We can write $y'_i = \frac{v(M)}{M} + \epsilon_i$, where $\sum_{i \in \mathcal{M}} \epsilon_i = 0$. We claim that, for all $S < M$, we can find $\mathcal{S}$ with cardinality S such that $\sum_{i \in \mathcal{S}} \epsilon_i \leq 0$. Clearly this is true for $S = M - 1$. By induction on S, it is sufficient to remove from $\{\epsilon_i\}_{i \in \mathcal{S}}$ its maximum element and its sum is still non positive. Hence, for all $S < M$, there exists $\mathcal{S} \subset \mathcal{M}$ such that $\sum_{i \in \mathcal{S}} y'_i = S\frac{v(M)}{M} + \sum_{i \in \mathcal{S}} \epsilon_i > v(S)$. Hence, $S\frac{v(M)}{M} > v(S)$ and the thesis is proven. ∎

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