Belief Propagation for Spatial Spectrum Access Games

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ABSTRACT

Consider a wireless network in which selfish users compete with each other for usage of spectrum. We view this competition as a spatial spectrum access game. There are two fundamental questions regarding this game: how to converge to the optimal Nash equilibrium, and how to converge fast. To answer these two questions, we apply a technique called Belief Propagation to design algorithms for users in this game, which can guarantee fast convergence to an optimal or nearly optimal pure strategy Nash equilibrium. Specifically, when the interference graph is an undirected tree or a directed acyclic graph, our algorithms can find the optimal Nash equilibrium in linear time. For general undirected interference graph, our algorithm converges fast as long as the game is potential (which is the case for many typical scenarios). For some other typical spatial spectrum access games, our algorithms can provide a good approximation to the optimal Nash equilibrium.

Categories and Subject Descriptors

C.2.1 [**Network Architecture and Design**]: Wireless Communication

Keywords

Wireless Network; Distributed Algorithm; Game Theory; Belief Propagation

1. INTRODUCTION

With the rapid development of mobile computing technology, the number of computing nodes in a network grows explosively while the number of available channels is an invariable natural resource limited by physical laws. Consider a multi-hop wireless network, e.g., a wireless ad-hoc network or a vehicular network. Since the nodes of the network usually belong to different users, these users will compete with each other for the usage of channels.

Naturally, we can view the above competition among users as a game, which is called *spatial spectrum access game* [5]. The players of this game are the selfish users, while the utilities of these users correspond to the network performance they experience. In principle, this spatial spectrum access game will converge to a Nash equilibrium eventually. Once the Nash equilibrium is reached, each user has no incentive to further change her own strategy, and thus a stable state of the entire network is achieved.

While this picture of converging to Nash equilibrium (stable state of network) looks nice, there are still two fundamental questions. First, a spatial spectrum access game can have multiple Nash equilibria, with completely different utitlities for users. How can users choose their strategies such that they have the best possible utilities in the finally reached Nash equilibrium? Second, the convergence to a Nash equilibrium can be a slow process. What strategies should users choose such that the entire network enter the stable state as quickly as possible? Our goal in this paper is to answer these two questions. That is, we would like to design distributed algorithms that converge *fast* to an *optimal* or near-optimal pure strategy Nash equilibrium in spatial spectrum access games.

In order to achieve this goal, we adopt the model of graphical game, in which the underlying graph is the interference graph, which captures the interference relations among users. We focus on the pure strategy Nash equilibrium (PSNE) because the probabilistic behavior of users can be hard-coded into the space of pure strategies. Our first step is to reduce the task of finding an optimal PSNE of a graphical game to the *maximum a posteriori* (MAP) estimation in a graphical model. We then use a modified version of *max-product belief propagation* (BP) algorithm to find the optimal PSNE. The BP algorithm has the following virtues:

1. BP is simple and easy to implement. We only need a single control channel for message passing.

2. BP is flexible so that the utility function of each user can be locally defined and needs not to be known by other users, satisfying privacy requirement. The final outcome of the PSNE reflects the utilities of all users.

3. The max-product BP is a greedy algorithm maximizing the utility of each user in every round, which is consistent with the selfishness of users. Moreover, each user only has local information. None of them could have enough information to be cheating.

Our Results In a general model (which can be used for, e.g., FDMA, TDMA, CDMA and CSMA/CA), we show the spatial spectrum access games on undirected interference graphs are potential games, guaranteeing the convergence to PSNE. We give a general reduction from the problem of finding optimal PSNE of graphical games to the MAP estimation in graphical model. The algorithmic results we obtain include:

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1. When the interference graph is an undirected tree or a directed acyclic graph, max-product belief propagation converges to the optimal PSNE of graphical games in linear time.

2. For any undirected interference graph, the BP-guided local dynamics converges to a PSNE as long as the game is potential, and experiments show that the convergence is fast for the considered protocols.

3. For some typical spatial spectrum access games, including the weighted q-state independent set, weighted q-coloring where $q > \Delta$, where q is the number of channels and Δ is the maximum degree of the interference graph, there is a *polynomial-time approximation scheme* (PTAS) for finding the optimal solution on planar graphs.

4. For k-outerplanar graphs where k is any fixed constant, there is a linear-time dynamic programming algorithm for finding the optimal PSNE for any spatial spectrum access games.

Related Work Mechanism design for spatial spectrum access has been studied extensively for different protocols [8, 13, 18]. This problem was modeled as various combinatorial problem such as independent set [19, 28], coloring [15, 23, 30] and matching [19, 28]. Designing spectrum sharing mechanism in the context of game theory has been studied in [5, 10, 13, 15, 22]. Static game was used to analyze channel assignment for single collision domain in [10]. Channel assignment for multi-collision domain by game theory was studied in [13, 22, 5]. Graphical game was first applied to address spatial spectrum access on undirected interference graphs in [22]. Then in [3] the model was generalized to directed interference graphs on which the graphical congestion game with linear latency function was studied. In [5], the concept of spatial spectrum access games was proposed and the convergence of real network protocols was answered for the first time. All the previous works on spatial spectrum access games focused on searching for a PSNE rather than finding the optimal PSNE for spatial spectrum access.

The graphical game was first studied in [20]. The connection between graphical models and graphical games was first discussed in [7], which was used to deciding the existence of PSNE on graphs with bounded treewidth.

Belief propagation is rediscovered many times in the history and has been widely used in artificial intelligence, information theory, statistical physics and many other related fields. In [26], they were developed for exact probabilistic inference in acyclic Bayesian networks. Much earlier in [11], they were used for decoding low-density parity-check (LDPC) codes. Recently, max-product and min-sum belief propagation were used to design various message-passing algorithms for combinatorial optimization problems, such as min-cost flow [12], maximum weight matching [2], maximum weight independent set [27]. All these optimization problems are special cases of optimization for PSNE of graphical games.

2. SYSTEM MODEL

We consider a wireless network with n heterogeneous users $V = \{1, \ldots, n\}$ and q orthogonal channels $Q = \{1, \ldots, q\}$. Each user is a transmitter with multiple radios and can access multiple channels simultaneously. In a fading environment, each user v_i has a limited interference range whose diameter is denoted by r_i. The distance between user v_i and user v_j is denoted by d_{ij}. The interference relations among all users are described by an interference graph $G = (V, E)$, with each vertex $v \in V$ representing a user in the network, and a directed edge going from u to v if and only if $r_u > d_{uv}$. For each user v, we use $N_{in}(v) = \{u : (u, v) \in E\}$ and $N_{out}(v) = \{u : (v, u) \in E\}$ to denote the set of in-neighbors and out-neighbors of v respectively, and $d_{in}(v) = |N_{in}(v)|$ and $d_{out} = |N_{out}(v)|$ the in-degree and out-degree of v respectively. If for any two users $u, v \in V$, when u can interfere with v, v can also interfere with u, the interference graph becomes an undirected graph. The directed edges (u, v) and (v, u) can be replaced by a single undirected edge $\{u, v\}$ and $N_{in}(v) = N_{out}(v) = N(v)$ for all $v \in V$. For each user i, let f_i denote her utility function, which maps the channel assignment to payoff. The median access problem is modeled as a graphical game on interference graph with these utility functions, where the notion of graphical game is to be formally defined later. We consider the utility functions realized by the MAC protocols extensively used in practice, including the contention-based protocol CSMA/CA, and the contention-free protocols FDMA, TDMA and CDMA.

1. For FDMA, TDMA and CDMA, a channel is divided equally into frequency bands, time slots or orthogonal codes. A strategy of user $i \in V$ is a boolean vector $s_i = (s_{i,1}, \ldots, s_{i,q})$ where $s_{i,j}$ indicates whether user i is using channel j. And $\sum_{j=1}^{q} s_{i,j} \le \delta_i$ where δ_i is the number of radios of user i. The utility function f_i is defined as in [10, 13]:
$$f_i = \sum_{j \in Q} \theta_j \frac{s_{i,j}}{s_{i,j} + \sum_{k \in N_{in}(i)} s_{k,j}},$$
where θ_j is the rate of channel j, which is assumed to be shared equally by the users.

2. For the CSMA/CA protocol, strategies of each user $i \in V$ are samely defined as above. When a channel j is sensed busy, user i makes a delay of time $\tau_{i,j}$ uniformly generated from $[0, \lambda_j]$. The utility function f_i is defined as in [5]:
$$\begin{aligned} f_i &= \sum_{j \in Q} s_{i,j} \int_0^{\lambda_j} \frac{\theta_j}{\lambda_j} \prod_{k \in N_{in}(i)} \left(\Pr[\tau_{k,j} > \tau_{i,j}] \right)^{s_{k,j}} \mathrm{d}\tau_{i,j} \\ &= \sum_{j \in Q} s_{i,j} \int_0^{\lambda_j} \frac{\theta_j}{\lambda_j} \left(\frac{\lambda_j - x}{\lambda_j} \right)^{p_j} \mathrm{d}x, \end{aligned}$$
where $p_j = \sum_{k \in N_{in}(i)} s_{k,j}$ and θ_j is the rate of channel j.

For time-varying channels, θ_j may be the expectation value of a stationary stochastic process.

Note that for all the considered protocols, we do not need to consider the case that two radios of the same user compete the usage of the same channel simultaneously.

In interference graphs, a directed cycle of length 2 is formed by two directed edges $(u, v), (v, u) \in E$ between the same pair of vertices. And we have the following simple proposition.

PROPOSITION 1. *In any directed interference graph, if there is no directed cycle of length 2, then there is no directed cycle of length greater than 2.*

PROOF. If $(u, v) \in E$ and $(v, u) \notin E$, then we have $r_u > d_{uv}$ and $r_v < d_{uv}$, and hence $r_u > r_v$. Let G be an interference graph with no directed cycle of length 2. If there is a directed cycle $L = (u_1, u_2, \ldots, u_\ell, u_1)$ with $\ell > 2$ in G, then we have $r_{u_1} > r_{u_2} > \ldots > r_{u_\ell} > r_{u_1}$, which leads to a contradiction. □

By this proposition, in our model the interference graph as shown in Figure 1 will never appear. Therefore, it is safe to characterize all directed acyclic interference graphs by the following definition.

DEFINITION 2. *An acyclic interference graph is a directed interference graph with no cycle of length 2.*

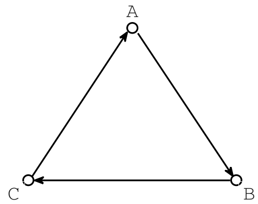

Figure 1: An Impossible Directed Interference Graph

3. PRELIMINARIES

3.1 Graphical games

The classic graphical game [20] is defined on undirected graphs. Here we consider the more general setting of directed graphs [3, 5].

DEFINITION 3. *(Graphical Game) A graphical game is a pair* (G, F) *where* $G = (V, E)$ *is a directed or undirected graph over* $V = \{1, \ldots, n\}$*, and* $F = \{f_i : i \in V\}$ *is a set of local payoff functions. Each node* $i \in V$ *represents a player whose in-neighborhood is defined as* $N_{in}(i) = \{j : (j, i) \in E\}$ *and strategy set is defined as* S_i*. For any joint action* $s = (s_1, \ldots, s_n) \in S = S_1 \times \ldots \times S_n$*, the payoff of i is given by* $f_i(s_i, s_{N_{in}(i)})$ *where* $s_{N_{in}(i)}$ *is the strategy profile of its neighbors.*

Note that f_i can be written as $f_i(s_i, s_{N_{in}(i)})$ as long as we consider only symmetric functions.

DEFINITION 4. *The pure strategy Nash equilibrium (PSNE) of a graphical game* (G, F) *is defined as a strategy profile* $s = (s_1, \ldots, s_n)$ *that for each* s_i *where* $i \in V$, $s_i = \text{argmax}_{y \in S_i} f_i(y, s_{N_{in}(i)})$.

DEFINITION 5. *A graphical game is called an ordinal potential game if there exist a potential function* $\Phi : S \to \mathbb{R}^+$ *such that when player i changes its strategy* s_i *to* s_i', $\text{sgn}(\Phi(s) - \Phi(s')) = \text{sgn}(f_i(s) - f_i(s'))$ *where* $s = (s_1, \ldots, s_i, \ldots, s_n)$ *and* $s' = (s_1, \ldots, s_i', \ldots, s_n)$.

DEFINITION 6. *A graphical game is called an exact potential game if there exist a potential function* $\Phi : S \to \mathbb{R}^+$ *such that when player i changes its strategy* s_i *to* s_i', $\Phi(s) - \Phi(s') = f_i(s) - f_i(s')$ *where* $s = (s_1, \ldots, s_i, \ldots, s_n)$ *and* $s' = (s_1, \ldots, s_i', \ldots, s_n)$.

DEFINITION 7. *A graphical game is called a weighted potential game if there exist a potential function* $\Phi : S \to \mathbb{R}^+$ *such that when player i changes its strategy* s_i *to* s_i', $\Phi(s) - \Phi(s') = w_i(f_i(s) - f_i(s'))$ *where* $s = (s_1, \ldots, s_i, \ldots, s_n)$ *and* $s' = (s_1, \ldots, s_i', \ldots, s_n)$.

It is well-known that if a game is potential it must possess a PSNE [24]. The PSNE can be achieved by the local improvement of strategies as long as the potential function exists [29]. When the potential function is maximal or minimal, the strategy profile is a PSNE. This is known as the *finite improvement property* (FIP).

3.2 Factor graphs

A factor graph [21] is one type of graphical model $\mathcal{G} = (X, \mathcal{F}, \mathcal{E})$, which is a bipartite graph where $X = \{X_1, \ldots, X_k\}$ is the set of variable nodes, with each X_i being a random variable whose joint distribution to be specified later, and $\mathcal{F} = \{F_1, \ldots, F_\ell\}$ is the set of function nodes, and $\mathcal{E} = \{\{X_i, F_j\} : i \in [k], j \in [\ell]\}$ is the set of edges adjoining variable nodes and function nodes. The bipartite graph $\mathcal{G}$ represents a function $g(x_1, \ldots, x_n) = \prod_{j=1}^m F_j(x_{\partial j})$ where $x_{\partial j}$ denotes the restriction of $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ on the neighborhood $\partial j = \{i : \{X_i, F_j\} \in \mathcal{E}\}$ of j in $\mathcal{G}$. We assume that F_j depends only on those variables x_i with $i \in \partial j$ so that g is well-defined. This gives rise to a natural probability distribution, in which the joint probability of any $\boldsymbol{x} \in X_1 \times \ldots \times X_n$ is defined as

$$P(\boldsymbol{x}) = \frac{1}{Z} \prod_{j=1}^{m} F_j(x_{\partial j})$$

where the normalizing factor $Z = \sum_{\boldsymbol{x}} \prod_{j=1}^m F_j(x_{\partial j})$ is called the *partition function*.

A compelling reason for adopting factor graph is because of its simplicity and flexibility for analysis. When the environmental constraints change, the factor graph can be easily modified to adapt to the new environment by changing function nodes. For instance, in cognitive radio networks, secondary users cannot use a channel which is being used by a primary user. Thus once an idle channel c is occupied by a primary user, all the function nodes associated with variable nodes corresponding to a secondary user interfering with the primary user can be changed to new function nodes by each combining with a new function node whose constraint function is a Boolean indicator function indicating that $x_i \neq c$ for all involved variables.

3.3 The reduction

Given a graphical game $(G, \mathcal{F})$ where $G = (V, E)$, a strategy profile $s = (s_1, \ldots, s_n)$ is a PSNE if and only if for all $i \in V$, s_i is the best response for $s_{N_{in}(i)}$. Hence a graphical game $(G, \mathcal{F})$ can be represented by a factor graph $\mathcal{G} = (V, F, \mathcal{E})$ in such a way that for each player $i \in V$, we have a variable node in the factor graph which takes the strategy s_i of play i as value, and for each player $i \in V$, we also have a function node F_i adjacent to the variable nodes corresponding to players in $N_{in}(i)$, including player i itself. The function F_i is defined as

$$F_i = \begin{cases} \exp(f_i) & \text{if } s_i = \text{argmax}_{y \in S_i} f_i(y, s_{N_{in}(i)}), \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Then searching the optimal pure strategy Nash equilibrium with maximum total payoff is equivalent to finding an assignment of variables with maximum product value in the factor graph, as follows:

$$\max \sum_{i=1}^{n} f_i = \max \left(\exp \left(\sum_{i=1}^{n} f_i \right) \right) \Leftrightarrow \max \prod_{i=1}^{n} F_i.$$

If the utility f_i is interpreted as "cost" and each player strives to minimize her cost, then F_i can be set as the dual version of max-product, that is

$$F_i = \begin{cases} \exp(-f_i) & \text{if } s_i = \text{argmax}_{y \in S_i} f_i(y, s_{N_{in}(i)}), \\ 0 & \text{otherwise.} \end{cases}$$

Then searching the optimal pure strategy Nash equilibrium with minimum total payoff is still equivalent to

$$\min \sum_{i=1}^{n} f_i = \max \left(\exp \left(- \sum_{i=1}^{n} f_i \right) \right) \Leftrightarrow \max \prod_{i=1}^{n} F_i.$$

Hence both utility maximization and cost minimization in graphical games can be reduced to computing the *maximum a posteriori* (MAP) value in corresponding factor graphs.

Sometimes we need to make a tradeoff between utility maximization and fairness. In this case, the constraint function F_i can

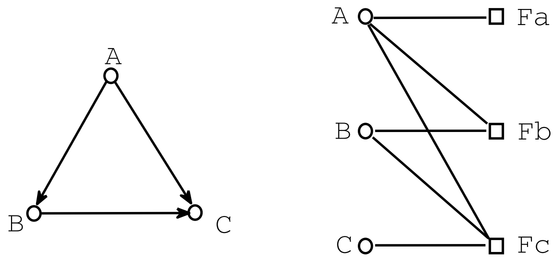

Figure 2: Factor Graph for Directed Acyclic Graph

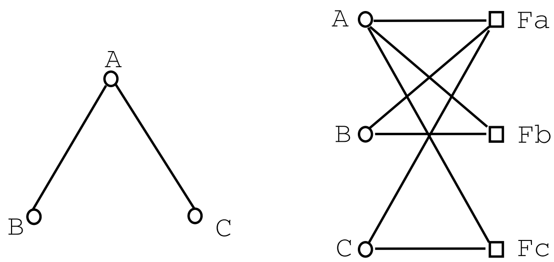

Figure 3: Factor Graph for Undirected Tree

be simply set as

$$F_i = \begin{cases} f_i = \exp(\log f_i) & \text{if } s_i = \operatorname{argmax}_{y \in S_i} f_i(y, s_{N_{in}(i)}), \\ 0 & \text{otherwise.} \end{cases}$$

By the inequality of arithmetic and geometric means, $n\left(\prod_{i=1}^n f_i\right)^{1/n}$ provides a lower bound for $\sum_{i=1}^n f_i$ and the equality holds if and only if $f_1 = \ldots = f_n$, which means the maximization of $\prod_{i=1}^n F_i$ (equivalently, $\prod_{i=1}^n f_i$) provides a tradeoff between utility maximization and fairness. We call this algorithm *"fairness propagation"*. However, this algorithm can only be used in a limited amount of cases where the maximization of $\prod_{i=1}^n F_i$ can be exactly and efficiently computed. In other cases the performance of this algorithm cannot be well measured.

3.4 Belief Propagation

A tutorial of classic BP and max-product BP can be found in [21]. By our reduction, the factor graph translated from a directed acyclic interference graph may not always be a tree. An example is shown in Figure 2. In addition, even for the undirected interference graph which is a tree, the resulting factor graph may not be a tree either, as shown in Figure 3.

We modify the standard max-product BP algorithm to deal with this complication. Let $\mu^{(t)}_{a\to i}$ denote the message sent from function node a to variable node i in t-th iteration and $x_{\partial a}$ denote the joint configuration of the neighbors (variable nodes) of a when the strategy s_j is fixed to be x_j for every player j. If a is F_i, then

$$\mu^{(t)}_{a\to i}(x_i) = \max_{x_{\partial a \setminus i}} \Big\{ F_a(x_{\partial a}) \prod_{j \in \partial a \setminus i} \mu^{(t)}_{j \to a}(x_j) \Big\}.$$

Otherwise, $\mu^{(t)}_{a\to i}(x_i) = 1$. Let $\mu^{(t)}_{i\to a}$ denote the message sent from variable node i to function node a in t-th iteration. Then

$$\mu^{(t+1)}_{i\to a}(x_i) = \prod_{b \in \partial i \setminus a} \mu^{(t)}_{b\to i}(x_i) = \mu^{(t)}_{F_i \to i}(x_i).$$

Each message is a key-value map where the key is x_i and the value is $\mu_{i\to a}(x_i)$ or $\mu_{a\to i}(x_i)$. From now on, when mentioning BP or max-product BP equations, we refer to this modified version.

Observe that if in the graphical game PSNE does not exist, then in the factor graph the messages $\mu_{F_i\to i}(x_i) = 0$ for all $x_i \in s_i$. Therefore, the fixed point representing the nonexistence of PSNE is

$$\mu_{i\to a}(x_i) = \begin{cases} 1 & \text{if } a \text{ is } F_i, \\ 0 & \text{otherwise,} \end{cases} \quad \mu_{a\to i}(x_i) = \begin{cases} 0 & \text{if } a \text{ is } F_i, \\ 1 & \text{otherwise.} \end{cases}$$

The belief propagation algorithm on factor graphs can be easily translated back to distributed algorithm on interference graphs in the following way. Since for every graphical game, the corresponding factor graph has the same number of variable node and function node, the function node F_i can be treated as the decision making process of player i. Therefore the messages exchanged between i and F_i will not appear in the network. The messages exchanged from variable node i and F_i to variable node j and F_j are the messages exchanged between player i and player j. By this translation, we have the distributed algorithm in the network.

4. THE EXISTENCE OF PSNE

We show that in undirected interference graphs, for the spatial spectrum access games using considered protocols, Nash equilibrium always exists, by showing that the games are potential.

LEMMA 8. *All spatial spectrum access games on undirected graphs using FDMA, TDMA and CDMA are potential games.*

PROOF. We transform throughput maximization to delay minimization where the delay of user i using channel j is defined as $s_{i,j} + \sum_{k \in N(i)} s_{k,j}$ if $s_{i,j} = 1$, which is a linear function of the number of neighbors of user i using channel j. When the delay function is linear, by the main result of [3], the spatial spectrum games using FDMA, TDMA and CDMA are potential games. □

When the utility function is $f'_i = \log f_i$, it is proved in [5] that spatial spectrum access games on undirected interference graph using CSMA/CA are ordinal potential games.

LEMMA 9. *All spatial spectrum access games on undirected graphs using CSMA/CA are potential games.*

PROOF. A PSNE for f'_i is also a PSNE for f_i since $\log f_i(s_1, \ldots, s_i, \ldots, s_n) \geq \log f_i(s_1, \ldots, s'_i, \ldots, s_n)$ for $\forall i \in V$ implies $f_i(s_1, \ldots, s_i, \ldots, s_n) \geq f_i(s_1, \ldots, s'_i, \ldots, s_n)$ for $\forall i \in V$. □

Since a potential game always has PSNE [24], we have the following theorem.

THEOREM 10. *For spatial spectrum access games on undirected graphs using FDMA, TDMA, CDMA, and CSMA/CA protocols, PSNE always exists.*

When the interference graph is directed, for $(i, j) \notin E$ and $(j, i) \in E$, we have $\frac{\partial^2 f_i}{\partial s_i \partial s_j} \neq 0$ and $\frac{\partial^2 f_j}{\partial s_i \partial s_j} = 0$, thus the ratio between them is $\pm\infty$. By Theorem 4.5 in [24], the potential function does not exist. This shows an evidence that for directed interference graphs, PSNE of spatial spectrum access game may not always exist.

We can use a similar local message-passing algorithm called *survey propagation* [4] to decide whether a PSNE exists. As in Section 3.3, the graphical game is translated into a *constraint satisfaction problem* (CSP), with constraints set as

$$F_i = \begin{cases} 1 & \text{if } s_i = \operatorname{argmax}_{y \in S_i} f_i(y, s_{N_{in}(i)}), \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Then the PSNE exists if and only if there exists a truth assignment for the CSP.

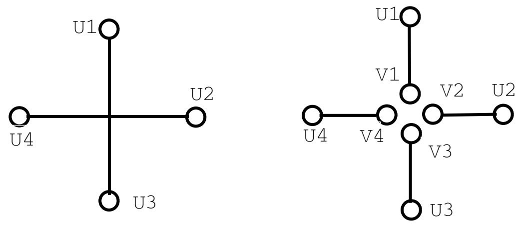

Figure 4: Reduction by Crossing Replacement

5. COMPUTATIONAL COMPLEXITY

In this section, we explore the computational complexity of graphical games to see how hard the problem is in general. It is known that deciding the existence of PSNE for graphical games is NP-complete [14]. The following theorem says the problem remains hard even on planar graphs.

THEOREM 11. *Deciding the existence of PSNE for graphical games on planar graphs is NP-complete.*

PROOF. This can be proved by a reduction from computing PSNE for graphical games $(G, \mathcal{F})$ on general graphs to computing PSNE for graphical games $(G', \mathcal{F}')$ on planar graphs, which is constructed by simply replacing each crossing with 4 new additional player whose payoff is always 0 and whose strategy will not affect other players' payoff, as shown in Figure 4. The strategies of additional player satisfy $s_{V_1} = s_{U_3}$, $s_{V_2} = s_{U_4}$, $s_{V_3} = s_{U_1}$ and $s_{V_4} = s_{U_2}$, then $(G, \mathcal{F})$ has a PSNE only if $(G', \mathcal{F}')$ has a PSNE. This completes the reduction. □

Since graphical congestion game [3] is a spacial case of graphical game, by the PLS-completeness result for classic congestion games [9], which are graphical congestion games on complete graph, we have the following theorem.

THEOREM 12. *Computing PSNE for graphical games is PLS-complete.*

On the other hand, the decision problem becomes trivial on directed acyclic graphs.

THEOREM 13 (CHEN *et al.* [5]). *There always exists a PSNE for graphical games on directed acyclic graphs.*

The PSNE of graphical games on directed acyclic graphs can be found in $O(|V| + |E|)$ time by topological sorting.

Observing that load balancing is a special case of graphical congestion game, by the NP-completeness of load balancing, the following theorem holds.

THEOREM 14. *Given a graphical game defined on an undirected graph and an integer $k > 0$, determining whether there exists a PSNE with cost at most k is NP-complete.*

The following theorem can be proved by the same reduction used in the proof of Theorem 11.

THEOREM 15. *Given a graphical game defined on an undirected* planar *graph and an integer $k > 0$, determining whether there exists a PSNE with cost at most k is NP-complete.*

Indeed, graphical game is a general model which covers many combinatorial optimization problems. When the utility function is properly set, as in Equation (3), the maximum independent set (MIS) problem can be coded as finding optimal PSNE in a special class of graphical games. Another observation is that the maximum weight q-coloring with $q > \Delta$ can also be reduced to optimizing PSNE of graphical games. It is known that MIS is MAX SNP-complete [25] and NP-hard to approximate within polynomial factor in polynomial time [16], which leads to the following inapproximability results.

THEOREM 16. *Computing optimal PSNE of graphical games is MAX SNP-complete.*

THEOREM 17. *For any $\epsilon > 0$, there does not exist any $n^{\epsilon-1}$-approximation algorithm for computing optimal PSNE of graphical games unless P = NP.*

PROOF. Both theorems can be proved by reducing maximum independent set to finding optimal PSNE in graphical games. We can set f_i as

$$f_i = \begin{cases} 1 & \text{if } s_{i,j} = 1 \text{ and } \forall j \in N(i),\ s_j = 0, \\ 0 & \text{if } s_{i,j} = 0 \text{ and } \exists j \in N(i),\ s_j = 1, \\ -1 & \text{if } s_{i,j} = 1 \text{ and } \exists j \in N(i),\ s_j = 1. \end{cases} \quad (3)$$

Then each PSNE is a maximal independent set and the optimal PSNE is a maximum independent set. Then by the inapproximability of maximum independent set, the theorem follows. □

6. ALGORITHM FOR ACYCLIC GRAPHS

In [20], an algorithm called "TreeNash" was proposed to compute a PSNE on undirected trees. Now we show that when the graphical game is defined on directed acyclic graphs or undirected trees, our modified max-product BP converge very fast to the fixed point $s = (s_1, \dots, s_n)$ maximizing the total payoff $\sum_{i=1}^n f_i$. Indeed, on these graphs, the max-product BP is equivalent to a parallel dynamic programming.

THEOREM 18. *For a graphical game on an undirected tree, suppose t^* is the diameter of the tree,*

1. *The max-product BP equations converge to a fixed point after $t > t^*$ iterations. That is, for any edge (i, a) in the factor graph and any $t > t^*$, $\mu_{i\to a}^{(t)} = \mu_{i\to a}^{(t^*)}$ and $\mu_{a\to i}^{(t)} = \mu_{a\to i}^{(t^*)}$.*
2. *The convergent fixed point is an optimal PSNE: for any player i and any $t > t^*$, $\mu_i^{(t)}(x_i) = \prod_{a\in\partial i} \mu_{a\to i}^{(t-1)}(x_i) = \max_{s:s_i=x_i} \prod_i F_i = \max_{s:s_i=x_i} \exp\left(\sum_i f_i\right)$.*

PROOF. We set the function associated to the function nodes as in Equation (1). Initially, for each player i, the message $\mu_{i\to a}^{(1)}(x_i) = 1$ if $s_i = x_i$. Otherwise $\mu_{i\to a}^{(1)}(x_i) = 0$. The proof is by induction on the depth of the tree. The base step of the induction is for the case of single-node graph whose factor graph contains one variable node and one function node connected by a single edge. We define the message from empty set is 1. In the first round, the message from F_i to i is

$$\mu_{F_i\to i}^{(1)}(x_i) = \begin{cases} F_i(x_i) & \text{if } x_i \text{ is the best strategy,} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that the messages have converged and $F_i(x_i) = \max\left(\exp(\sum_i f_i)\right)$ is the maximum total utility of the graphical game on the single-node tree. Then $\mu_{i\to F_i}^{(t)}(x_i) = 1$ and $\mu_{F_i\to i}^{(t)}(x_i) = \mu_{F_i\to i}^{(1)}(x_i)$ for any $t > 1$. Hence the messages have converged.

Suppose it holds for tree with depth τ, then we show that it also holds for tree with depth $\tau+1$. Consider the tree rooted at i of depth is $\tau+1$, whose the subtree is of depth at most τ. Then after at most τ rounds, $\mu_{j\to F_i}(x_j)$ where $j\in\partial F_i\backslash i$ is the maximum total utility for subtree T_j rooted at j whose configuration is x_j. Then we have

$$\mu^{(\tau)}_{F_i\to i}(x_i)=\max_{\partial F_i\backslash i}\left\{F_i(x_{\partial F_i})\prod_{j\in\partial F_i\backslash i}\mu^{(\tau)}_{j\to F_i}(x_j)\right\}.$$

We can see that $\mu_i^{(\tau+1)}(x_i)=\prod_{a\in\partial i}\mu^{(\tau)}_{a\to i}(x_i)=\mu^{(\tau)}_{F_i\to i}(x_i)$ is the maximum total utility of graphical game when s_i is fixed to x_i. This complete the induction. □

THEOREM 19. *For a graphical game on a directed acyclic interference graph, suppose t^* is the length of the longest path from any of the node whose in-degree is 0 to any of the node whose out-degree is 0. Such nodes always exist since the interference graph is acyclic.*

1. *The max-product BP equations converge to a fixed point after $t>t^*$ iterations. That is, for any edge (i,a) in the factor graph and any $t>t^*$, $\mu^{(t)}_{i\to a}=\mu^{(t^*)}_{i\to a}$ and $\mu^{(t)}_{a\to i}=\mu^{(t^*)}_{a\to i}$.*
2. *The convergent fixed point is an optimal PSNE: for any player i and any $t>t^*$, $\mu_i^{(t)}(x_i)=\prod_{a\in\partial i}\mu^{(t-1)}_{a\to i}(x_i)=\max_{s:s_i=x_i}\prod_i F_i=\max_{s:s_i=x_i}\exp\left(\sum_i f_i\right)$.*

PROOF. This theorem is proved by a structural induction on the depth of the directed interference graph. The depth of node i is defined as the maximum length of shortest paths from any node whose in-degree is 0. The depths of the nodes with in-degree 0 are set as 0.

It is easy to see that the function nodes of variable nodes with depth larger than 0 will only sends "1" messages to the variable nodes with depth 0, which will not affect their computation of messages after round 0. Hence in round 0 the messages of nodes with depth 0 has converged. Similarly, the function nodes of variable nodes with depth larger than $t+1$ will only sends "1" messages to the variable nodes with depth t. Hence if in round t the messages of nodes with depth t converge, then the messages of nodes with depth $t+1$ will converge in round $t+1$. The computation details are similar to theorem 18. This completes the induction. □

7. ALGORITHM FOR CYCLIC GRAPHS

We study the convergence rate of our max-product BP algorithm to PSNE on cyclic undirected graphs. We start by showing an impossibility result.

THEOREM 20. *Synchronous max-product BP does not always converge for spatial spectrum access games on cyclic graphs.*

PROOF. We set the function associated to the function nodes as in Equation (2), with the initial condition $\mu^{(1)}_{i\to a}(x_i)=1$ if $s_i=x_i$ and $\mu^{(1)}_{i\to a}(x_i)=0$ if otherwise. Let $\mu^{(t)}_{i\to a}(x_i)$ and $\mu^{(t)}_{a\to i}(x_i)$ denote the messages sent from i to a and from a to i in the t-th iteration respectively. Then for $i\in V$, we have

$$\mu^{(1)}_{F_i\to i}(x_i)=\begin{cases}1, & \text{if } x \text{ is the best response of } s_{N(i)},\\ 0, & \text{otherwise.}\end{cases}$$

Hence it holds that

$$\mu^{(2)}_{i\to a}(x_i)=\begin{cases}1, & \text{if } a \text{ is } F_i,\\ \mu^{(1)}_{F_i\to i}(x_i), & \text{otherwise.}\end{cases}$$

We defined $s^{(t)}=\{s^{(t)}_1,\ldots,s^{(t)}_n\}$ where $s^{(t)}_i=x_i$ such that $\mu^{(2)}_{i\to a}(x_i)=1$ for $a\neq F_i$. Then for $i\in V$, we have

$$\mu^{(t)}_{F_i\to i}(x_i)=\begin{cases}1, & \text{if } x_i \text{ is the best response of } s^{(t)}_{N(i)},\\ 0, & \text{otherwise.}\end{cases}$$

Suppose the interference graph is an undirected complete graph of n vertices and $Q=\{1,2,3\}$. The rates of the channels satisfy $\theta_1=\theta_3>\theta_2$. All users initialize their strategies by randomly choosing channel from $Q'=\{1,2\}$. Then for $t\in\mathbb{N}^+$,

$$\mu^{(2t-1)}_{F_i\to i}(x_i)=\begin{cases}1, & \text{if } x_i=3\\ 0, & \text{otherwise,}\end{cases}\quad \mu^{(2t)}_{F_i\to i}(x_i)=\begin{cases}1, & \text{if } x_i=1\\ 0, & \text{otherwise.}\end{cases}$$

Then the BP process falls into a infinite loop and never converge. □

We can avoid this issue by introducing the concept of *BP-guided local dynamics* where the strategy update is guided by the *asynchronous max-product BP* equations:

$$\begin{aligned}\mu^{(t+1)}_{i\to a}(x_i) &= \mu^{(t)}_{F_i\to i}(x_i),\\ \mu^{(t)}_{F_i\to i}(x_i) &= \max_{\partial F_i\backslash i}\left\{F_i(x_{\partial F_i})\prod_{j\in\partial F_i\backslash i}\mu^{(\tau_j)}_{j\to F_i}(x_j)\right\}\end{aligned}$$

where $\mu^{(\tau_j)}_{j\to F_i}(x_j)$ is the τ_j-th and the latest message sent from j to F_i. By setting F_i as in Equation (2), $\mu^{(1)}_{i\to a}(x_i)=1$ if $s_i=x_i$ and $\mu^{(1)}_{i\to a}(x_i)=0$ if otherwise, then $\mu^{(t)}_{F_i\to i}(x_i)=1$ if x_i is the best response for $s_{N_{in}(i)}$. If the best response x_i computed by $\mu^{(t)}_{F_i\to i}(x_i)$ is not the current s_i, then user i modify its strategy to $s_i=x_i$.

For BP-guided local dynamics, the convergence time is measured by the number of strategy modification of all the users instead of the number of rounds. We further require that no two updates of strategies involving interfering users can occur simultaneously, then the following holds.

THEOREM 21. *The BP-guided local dynamics converges to a PSNE for spatial spectrum access games on undirected graphs if the game is potential.*

PROOF. For potential games, the utility improvement of each user will also improve the value of potential function. When the potential function is maximal, the strategy profile $s=(s_1,\ldots,s_n)$ is a PSNE after convergence. And by the finite improvement property, this algorithm will terminate in finite steps. □

By Theorem 12, computing a PSNE for graphical games is PLS-complete. Therefore, instead of exact PSNE we study the convergence rate to approximate PSNE, called ϵ-PSNE, defined in [6].

DEFINITION 22. *(ϵ-PSNE) A state $s=(s_1,\ldots,s_n)$ is an ϵ-PSNE if no player can improve its payoff by a factor greater than $\epsilon\in[0,1)$. If f_i is interpreted as cost, then $f_i(s_1,\ldots,s_i',\ldots,s_n)\geq(1-\epsilon)f_i(s_1,\ldots,s_i,\ldots,s_n)$ for all $s_i'\in S_i$. If f_i is interpreted as utility, then $f_i(s_1,\ldots,s_i',\ldots,s_n)\leq(1+\epsilon)f_i(s_1,\ldots,s_i,\ldots,s_n)$ for all $s_i'\in S_i$.*

To reach an ϵ-PSNE, a user will change its strategy if and only if it can improve its utility by a factor at least ϵ, which is called ϵ-move [6]. To study the rate of convergence to a ϵ-PSNE, we transform the utility maximization problem into its dual version, the cost minimization. The cost is defined as $C_i=\sum_{j\in s_i}C_{i,j}=\sum_{j\in s_i}1/f_{i,j}$ where $f_{i,j}$ is the utility of user i using channel j, which can be naturally defined for all our considered protocols. Note that $f_{i,j}$ is a function of the number of neighbors using channel j, we define α as the lower bound of a constant which satisfies $f_{i,j}(x)\leq\alpha\cdot f_{i,j}(x+1)$ for $x\in[n-1]$. Then $C_{i,j}(x+1)\leq\alpha\cdot C_{i,j}(x)$, satisfying the *$\alpha$-bounded jump* condition [6]. By this transformation, the spatial spectrum access games using FDMA, TDMA and CDMA becomes exact potential games, and hence possess a potential function Φ [3].

We show the convergence rates of BP-guided local dynamics which are assumed to satisfy the following conditions.

1. **Condition 1:** The player with maximum cost can make an ϵ-move.
2. **Condition 2:** If the player v with maximum cost cannot make an ϵ-move, at least one of its neighbors $u \in N(v)$ has cost greater than any player $u' \in N(u)\backslash v$ and can make an ϵ-move.

Let Δ denote the maximum degree of the interference graph and Φ denote the potential function of spatial spectrum access games. The upper bounds of convergence time for the dynamics satisfying conditions 1 and 2 are given by the following theorems.

THEOREM 23. *If a spatial spectrum access game possesses an exact potential function Φ and the dynamics satisfies the condition 1, then the convergence time is $O(n\epsilon^{-1}\log\Phi_{\max})$.*

PROOF. Since $\Phi(s) \leq \sum_i C_i(s)$ for any strategy profile s, then each time the player with maximum cost makes a ϵ-move, it decreases the potential function at least a factor $\frac{\epsilon}{n}$. Hence $(1-\frac{\epsilon}{n})^t\Phi_{\max} < \Phi_{\min}$, then $t = O(n\epsilon^{-1}\log\Phi_{\max})$. □

THEOREM 24. *If a spatial spectrum access game possesses an exact potential function Φ and the dynamics satisfies the condition 2, then the convergence time is $O(n\alpha^{\Delta}\epsilon^{-1}\log\Phi_{\max})$.*

PROOF. Suppose i is the user with maximum cost. For $(i, j) \in E$, let $s = (s_1, \ldots, s_i, \ldots, s_j, \ldots, s_n)$ and $s' = (s_1, \ldots, s'_i, \ldots, s_j, \ldots, s_n)$, $s'' = (s_1, \ldots, s_i, \ldots, s'_i, \ldots, s_n)$. If user j moves from s_j to s'_i, then $C_j(s'') \leq (1-\epsilon)C_j(s)$. Since user i did not move to s'_i, then $(1-\epsilon)C_i(s) \leq C_i(s')$. For each channel $e \in s'_i$, $C_{i,e}(s') \leq \alpha^{\Delta}C_{j,e}(s'')$, then $C_i(s') \leq \alpha^{\Delta}C_j(s'')$. Hence it holds that $(1-\epsilon)C_i(s) \leq C_i(s') \leq \alpha^{\Delta}C_j(s'') \leq \alpha^{\Delta}(1-\epsilon)C_j(s)$. The ϵ-move made by player j will decrease the potential function by a constant at least $\frac{\epsilon}{n\alpha^{\Delta}}$. Hence $(1-\frac{\epsilon}{n\alpha^{\Delta}})^t\Phi_{\max} < \Phi_{\min}$, then $t = O(n\alpha^{\Delta}\epsilon^{-1}\log\Phi_{\max})$. □

If the condition 1 and 2 to is loosed such that they are satisfied at least once respectively during T strategy updates, then the convergence time becomes $O(nT\epsilon^{-1}\log\Phi_{\max})$ and $O(nT\alpha^{\Delta}\epsilon^{-1}\log\Phi_{\max})$. This technique can be easily generalized to weighted potential games.

8. ALGORITHM FOR PLANAR GRAPHS

We give approximation algorithms for computing optimal PSNE in some typical spatial spectrum access games on planar graphs in the sense of *polynomial time approximation scheme*. This is useful because the interference graphs of wireless mesh networks are often planar.

DEFINITION 25. *A polynomial time approximation scheme (PTAS) for an optimization problem is an polynomial time algorithm $A(I, \epsilon)$ which takes an instance I of the problem and a parameter ϵ and always return a solution with accuracy $(1-\epsilon)$ for maximization and $(1+\epsilon)$ for minimization.*

We consider the spatial spectrum access games induced by independent sets or colorings studied in [19, 28, 15, 23, 30]. When transmission time is long and users can be blocked to avoid collisions, then the optimization of spatial spectrum access can be modeled as a q-state maximum weight independent set (q-MWIS) problem where the set of state is denote as $\{0, \ldots, q-1\}$. A user is in state $j \in [1, q-1]$ means the user is using channel j, and state 0 means the user is idle or blocked. The model of independent set for scheduling and congestion control in wireless networks has been widely studied [19, 28].

If the system requires high quality of service (QoS) which blocking user is not allowed, then the game can be modeled as a maximum weight q-coloring (q-MWC) problem as in [15, 23, 30], where the set of colors is $\{1, \ldots, q\}$. A user is of color j means the user is using channel j. Since deciding the existence of a proper coloring is nontrivial for $q \leq \Delta$ where Δ is the maximum degree of the graph, we only consider the case where $q > \Delta$. Both q-MWIS and q-MWC can be formulated as computing optimal PSNE of spatial spectrum access games. We have the following algorithmic result generalized from the PTAS for maximum independent set problem on planar graphs in [1].

THEOREM 26. *There exist PTAS for computing (q-state) maximum weight independent set and maximum weight q-coloring with $q > \Delta$ on planar graphs.*

Given a planar embedding of planar graph G, the level-1 vertices are the vertices on exterior face, and a vertex is a level-k if it is on the exterior face by deleting all the vertices of level $< k$. A planar graph is k-outerplanar if for some embedding it has no vertex of level $> k$.

THEOREM 27. *The optimal PSNE of graphical games can be exactly computed in $O(q^{O(k)} \cdot n)$ time for k-outerplanar graphs.*

The algorithm in this theorem is a dynamic programming algorithm, which can either be generalized from the dynamic programming in [1] for computing MIS on k-outerplanar graphs, or specialized from the dynamic programming in [31] for counting CSPs on graphs of bounded treewidth $O(k)$ (which cover k-outerplanar graphs). In particular, for q-MWIS and q-MWC, this exact algorithm can be combined into a PTAS on planar graphs using the same routine designed in [1] for maximum independent sets. This gives us the PTAS in Theorem 26. We omit detailed proofs here.

The next theorem shows that in general we may not expect to always have good approximation of PSNE as in the cases of q-MWIS and q-MWC.

THEOREM 28. *There is no PTAS for computing optimal PSNE for graphical games on planar graphs unless P = NP.*

PROOF. If the PTAS exists, then there is an algorithm can approximate the total payoff of PSNE within accuracy $(1-\epsilon)$ for any $\epsilon > 0$, which implies a polynomial time algorithm for deciding the existence of PSNE on planar graphs, which is NP-hard by Theorem 11. □

The following heuristics can be considered to cope with the above inapproximability result. Given a spatial spectrum access game on planar graph G, we can use the linear-time algorithm in [17] to generate a planar embedding. Fix a constant k. Applying the dynamic programming algorithm in Theorem 27, we can find an optimal PSNE for the k-outerplanar components separated by the level-ℓ vertices for all $\ell \mod (k+1) = 0$, and then randomly assign channels for the level ℓ vertices. Since the game is potential, we can then apply the asynchronous max-product BP to converge to a PSNE. The heuristic part is that we wish the randomly assigned channels would not distort too much from a globally optimal PSNE.

9. EVALUATIONS

We evaluate the performance of BP-guided local dynamics for spatial spectrum access games. We consider the case where the size of the network is $1000m \times 1000m$ and $q = 5$. The rates of channels is in the range from 10Mbit/s to 60Mbit/s. Each user has a interference range 50m. The locations of n users are uniformly distributed in the network. For each case, we generate a random interference graph and evaluate the performance for 11 rounds to test the robustness. Since the utility functions of FDMA, TDMA and CDMA has the same form, we only evaluate the performance of FDMA for the contention-free protocols, and the CSMA/CA for the contention-based protocol.

We first evaluate the performance of random dynamics on undirected interference graphs, which is BP-guided local dynamics where the order of users updating their strategies is random. We consider four cases where the number of users $n = 100, 200, 500, 1000$. For each case compare the total utility of users between random access and PSNE, as shown in Figure 5 and Figure 6. The corresponding convergence time which is measured by the number of total strategy updates is shown in Figure 7 and Figure 8. Since the PSNE may not exist on directed interference graphs, we only evaluate the performance for cases where n is relatively small. The performance is similar to those of undirected interference graphs as long as the dynamics can converge.

We can see that for all the communication protocols considered in this paper the BP-guided local dynamics can improve the total utility of users by more than 60%. And this improvement is very stable which does not decrease when the number of users increases from 100 to 1000. Moreover, the evaluation shows that the average number of strategy updates of each user is about 0.7-0.8, which is less than 1. This shows the BP-guided local dynamics can converge very fast to a PSNE of spatial spectrum access games. Thus the cost of spectrum mobility is very small.

We then consider the cases where the orders of users updating their strategies are not random. We consider six types of dynamics. The first three has been considered in [6] to study the convergence properties of congestion games where the orders are determined by global priorities, listed as follows:

1. Largest cost dynamics (LCD): the players with largest delay (minimum throughput) has priority to move, that is, $f_i(s) \leq f_j(s)$ for all $j \in V$.

2. Largest gain dynamics (LGD): the players with largest absolute improvement has priority to move, that is, $f_i(s') - f_i(s) \geq f_j(s'') - f_j(s)$ for all $j \in V$, where s' is the strategy profile after user i updates its strategy and s'' is the strategy profile after user j updates its strategy.

3. Largest ratio dynamics (LRD): the players with largest relative improvement has priority to move, that is, $\frac{f_i(s')-f_i(s)}{f_i(s)} \geq \frac{f_j(s'')-f_j(s)}{f_j(s)}$ for all $j \in V$, where s' is the strategy profile after user i updates its strategy and s'' is the strategy profile after user j updates its strategy.

The next three types of dynamics is similar to the above three, except the orders of strategy updates are determined by local priorities, listed as follows:

1. Local Largest cost dynamics (LLCD): the players with largest delay (minimum throughput) has priority to move, that is, $f_i(s) \leq f_j(s)$ for all $j \in N(i)$.

2. Local Largest gain dynamics (LLGD): the players with largest absolute improvement has priority to move, that is, $f_i(s') - f_i(s) \geq f_j(s'') - f_j(s)$ for all $j \in N(i)$, where s' is the strategy profile after user i updates its strategy and s'' is the strategy profile after user j updates its strategy.

3. Local Largest ratio dynamics (LLRD): the players with largest relative improvement has priority to move, that is, $\frac{f_i(s')-f_i(s)}{f_i(s)} \geq \frac{f_j(s'')-f_j(s)}{f_j(s)}$ for all $j \in N(i)$, where s' is the strategy profile after user i updates its strategy and s'' is the strategy profile after user j updates its strategy.

It is easy to see that all the users who have local priorities to move form an independent set of the interference graph. We do not require them to move simultaneously. From the evaluation we can see that for all the communication protocols considered in this paper, the performances of LGD (LLGD) and LRD (LLRD) are almost the same, which is similar to random dynamics that can improve the network performance by more than 60%, as shown in Figure 9. When n is small, the performance of LCD (LLCD) is very close to LGD (LLGD) and LRD (LLRD). However, as n increases, the performance of LCD (LLCD) decreases compared to LGD (LLGD) and LRD (LLRD). When $n = 1000$, for all the considered protocols, the LCD can only improve the network performance by 25%. The LLCD is better, when $n = 1000$ it can improve the network performance by about 50%, as shown in Figure 10.

Due to the space limit, we omit the evaluation of convergence time of these dynamics. Actually, the convergence time of LGD (LLGD) and LRD (LLRD) is almost the same for all the communication protocols considered in this paper, which is similar to their performances. For LCD (LLCD), although its performance is poorer, the convergence is faster. An empirical conclusion is, when the improvement is larger, the convergence time is longer. This provides a method to make a tradeoff between the performance and convergence.

10. CONCLUSION

We translate the problem of finding optimal pure strategy Nash equilibrium (PSNE) in spatial spectrum access games for channel assignment in wireless networks into estimating maximum a posteriori (MAP) assignment in factor graphs, and then propose to use a modified version of max-product belief propagation to solve the problem. Our algorithms are guaranteed to converge in linear time to an optimal PSNE if the interference graph is an undirected tree or directed acyclic interference graph, and converge fast to a near-optimal PSNE when the interference graph is undirected. Moreover, the optimization problem can be exactly solved on k-outerplanar graphs and for some typical special cases it has polynomial time approximation scheme on planar graphs. There are some interesting problems remaining open such as the approximability of finding the optimal PSNE for spatial spectrum access games on planar graphs or families of graphs arising from wireless networks.

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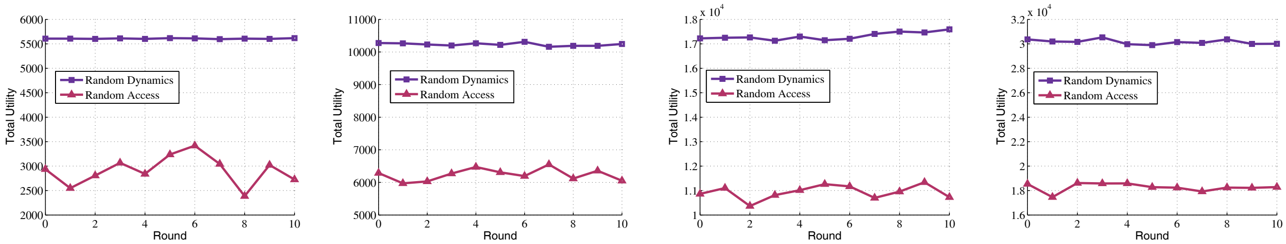

Figure 5: Performance evaluation for FDMA when n = 100, 200, 500, 1000

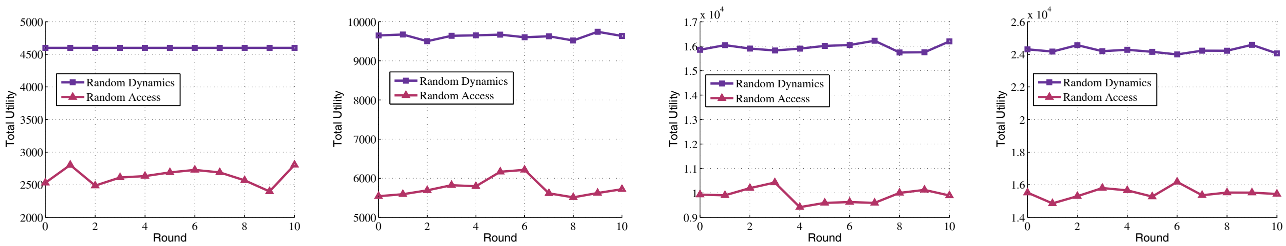

Figure 6: Performance evaluation for CSMA/CA when n = 100, 200, 500, 1000

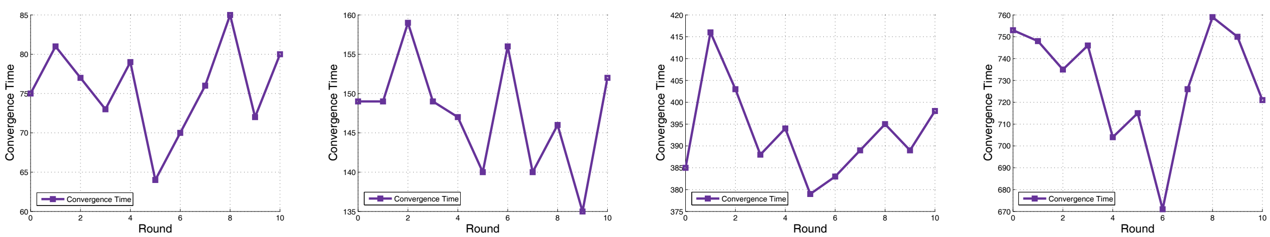

Figure 7: Convergence time for FDMA when n = 100, 200, 500, 1000

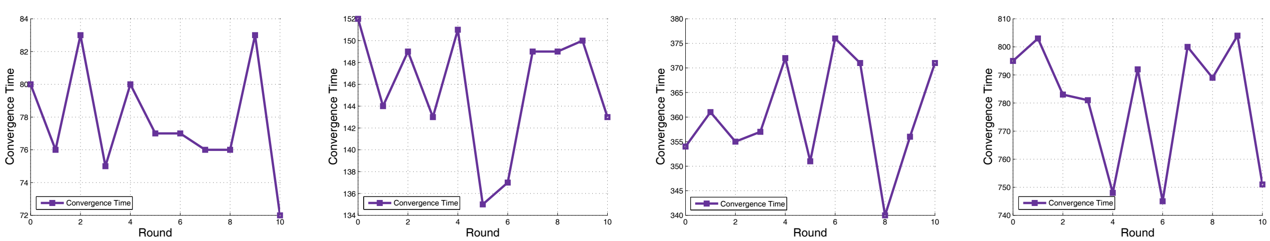

Figure 8: Convergence time for CSMA/CA when n = 100, 200, 500, 1000

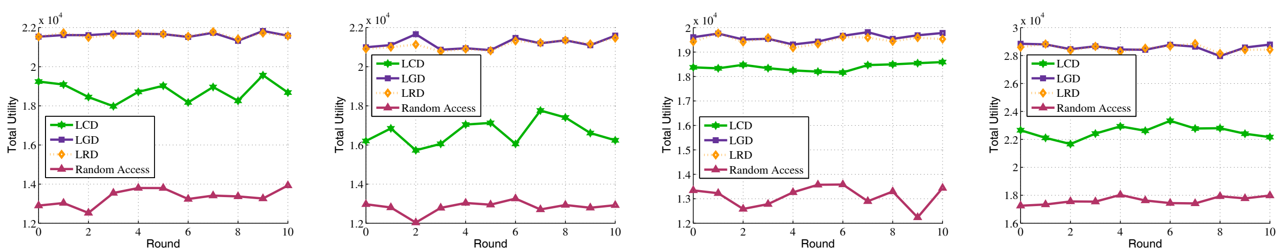

Figure 9: Performance evaluation of global-priority dynamics for FDMA and CSMA/CA when n = 500, 1000

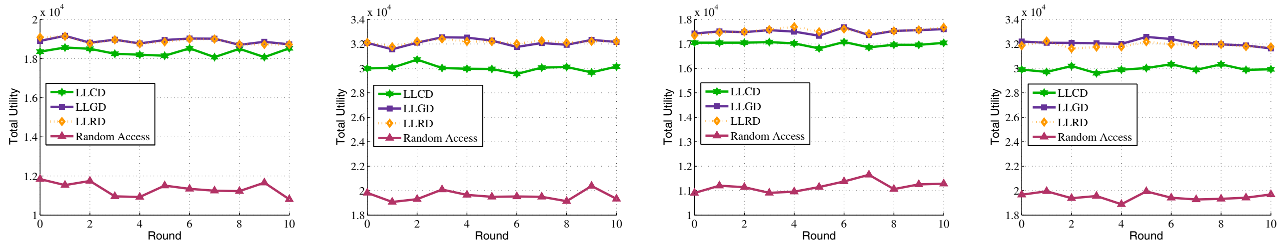

Figure 10: Performance evaluation of local-priority dynamics for FDMA and CSMA/CA when n = 500, 1000

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